Logics and Calculi for Quantum: An unexpected journey! (or how I came to meet Professor Luís Soares Barbosa)

Carlos Tavares

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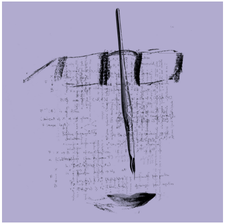

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...

Professor Luís: "What is your area of interest?", 11:19 PM

Me: "Quantum computing, more on complexity and etc", 11:39 PM

Professor Luís: "Oh... I will also be the director of the Physics Engineering degree, my interests lie aroud $C*$-algebras, *stochastic* coalgebras, quantum logic...", 11:50 PM

- **2015/2016:** Official start of thesis work;

  Thesis: "Foundations for quantum algorithms and complexity"

- **About logics and calculi:**

  A logic for the quantum assembly language (LQASM)

Standard quantum logic

- Introduced in 1936, by Von Neumman and Birkhoff.
- It is the logic of observable properties, in particular,

$$\sim\sim A = A, \text{ where } \sim A \text{ is the set of vectors orthogonal to A}$$

- Examples of observable properties: momentum, position.
- Observables do not generally commute:

$$[A, B] \neq 0 \text{ for A and B observables and } [\text{A, B}] = \text{A.B - B.A}$$

- Most famous example: *Heisenberg uncertainty principle*: $[position, momentum] \neq 0$

Standard quantum logic

- Commutation relations can be multiple-dimensional. Examples spins in x,z $[\sigma_x, \sigma_z]$ of singlet systems (total spin 0) involving two particles, leading to etangled states in two dimensional systems

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

- Observables have different commutation relations in each dimension

Standard quantum logic

Two consequences

- Distributivity law does not hold in quantum logic. Quantum logic "lives" in a non-distributive lattice
- Logic is different (i.e. rules valid in lattice) for each dimension. There is no way of combining lattices of smaller system to obtain larger ones, i.e. no tensor operator.
- Reasoning about quantum resources such as entanglement, cannot be done in a compositional way, dimensional-wise.

Challenges for proof theory

- Find models that possess tensor operations.
- Is it possible to have a complete set of axioms for quantum logic?
  - Finite dimesional systems: Yes, logic is decidable (infinite but recursively enumerabe set of axioms?).
  - Infinite dimensional systems: Unknown, logic is undecidable.

Dynamic quantum logic

- Introduced by Baltag and Smets in 2004;
- Retrieves quantum logic through relational structures (orthogonality corresponds to pairs of states where projectional tests are not a valid transition);
- Able to capture unitary transitions, i.e. quantum programs.

The QASM language

$$
\begin{array}{lcl}
\langle \pi_q \rangle & ::= & \text{x qreg\_id [index]} \mid \text{z qreg\_id [index]} \mid \text{h qreg\_id [index]} \mid \text{cx qreg\_id [index}_1\text{], qreg\_id [index}_2\text{]} \\
& \mid & \textbf{measure} \text{ qreg\_id [index]} \rightarrow \text{creg\_id [index]} \\
& \mid & \pi_q ; \pi_q \\
\langle \pi \rangle & ::= & \textbf{creg} \text{ id [size]} \mid \textbf{qreg} \text{ id [size]} \\
& \mid & \textbf{if} < \textit{test} > \textbf{then } \pi_q \\
& \mid & \pi ; \pi
\end{array}
$$

Figure: A fragment of the QASM programming language.

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 & \mid & \pi ; \pi
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Figure: A fragment of the QASM programming language.

Why bother?

- Classical information is fundamentally different than quantum information, the former accepts copies, the latter does not!
- Usually the natural semantics for this is given by density operators and superoperators. There are logics that follow this approach.
- We rather did an extension of a model of dynamic logic, the challenges are the modeling of measurements and if statements.

Logic for quantum assembly - Syntax

$$
\begin{array}{lcl}
\langle \varphi_q ? \rangle & ::= & id_i^q \mathbf{==} \text{ a, where } a \in \{1, 0\} \\
\langle \varphi ? \rangle & ::= & \varphi_q ? \\
\langle \pi_q \rangle & ::= & \mathbf{x}\ id_i^q \mid \mathbf{z}\ id_i^q \mid \mathbf{h}\ id_i^q \\
 & \mid & \mathbf{cx}\ id_i^q,\ id_i^q \\
\langle \pi \rangle & ::= & \pi ; \pi \\
\langle p \rangle & ::= & i_{r_i}^q \text{ with } i \in \{0, 1\} \\
\langle \varphi \rangle & ::= & \text{p} \mid \top \mid P^{\geq r} \varphi \mid [\pi]\ \varphi \mid \neg \varphi \mid \varphi \wedge \varphi
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\langle \varphi_c ? \rangle & ::= & \underline{id_i^c \; \mathbf{==} \; \mathrm{a}, \text{ where } a \in \{1, 0\}} \\
\langle \varphi ? \rangle & ::= & \underline{\varphi_c ?} \mid \varphi_q ? \\
\langle \pi_q \rangle & ::= & \mathbf{x} \; id_i^q \mid \mathbf{z} \; id_i^q \mid \mathbf{h} \; id_i^q \\
 & \mid & \mathbf{cx} \; id_i^q, \, id_i^q \\
 & \mid & \underline{\mathbf{meas} \; id_i^q \text{ to } id_i^c} \\
 & \mid & \pi_q ; \pi_q \\
\langle \pi \rangle & ::= & \underline{\mathbf{creg} \text{ id [size]} \mid \mathbf{qreg} \text{ id [size]}} \\
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Semantics

Labelled transition system

$$M = (S, [\![.]\!]_p : \mathcal{A}_p \to 2^S, [\![.]\!]_\pi : \mathcal{A}_\pi \to 2^{S \times S})$$

There is no tensor operation for models in quantum logic, i.e. for each dimension a different models and axioms apply.

State space S

$$\underbrace{\underbrace{\mathcal{H}^{2n} \times \mathcal{H}^{2n}}_{\text{quantum registers}} \times \ldots \times \underbrace{\mathcal{C}_i^2 \times \ldots \times \mathcal{C}_i^2}_{\text{classical register}} \times \ldots}_{S} .$$

$$\mathcal{C}^{2n} \equiv \mathcal{H}^{2n} \equiv \sum_{i \in \{0,1\}^n} \lambda_i \, |i\rangle \text{ where } \sum_i \lambda_i * \lambda_i^\dagger = 1$$

Semantics

Programs are pairs of states

$$[\![.]\!]_\pi : \mathcal{A}_\pi \to 2^{S \times S}$$

Some of them are not easily captured by unitary operators and tests:

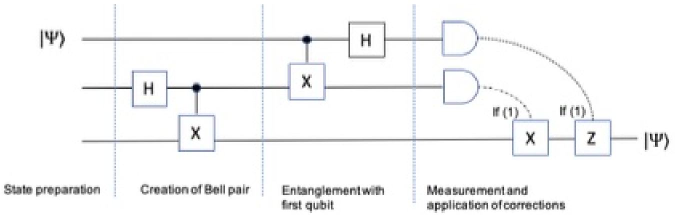
Measurements

$[\![$**meas** id_i^q to $id_i^c]\!] \simeq$ quantum test of $id_i^q \circ$ copy id_i^q to classical bit id_i^c

If statements

$[\![$**if** φ_c? **then** $\pi_q]\!] \simeq$ (program $\pi_q \circ$ quantum test of φ_c?) and statistics of φ_c? is maintained.

Proof of the teleportation protocol



[qreg q[3]; creg c[2]; $p_0(\alpha)$ q_0; h q_1^q; cx q_1^q, q_2^q; cx q_0^q, q_1^q; h q_0^q; meas q_0^q to c_0^c; meas q_1^q to c_1^c; if (c [0] $==$ 1) x q_2^q; if (c [1] $==$ 1) z q_2^q]

$\left(P^{=\alpha_1} 0_2^q \wedge P^{=1-\alpha} 1_2^q\right)$,

Decidability

Technique: Showing that there is a *computable reduction* of every grammatic generator to the first order theory of closed fields (FOR), known to be decidable by Tarski.

Main ideia: There is an equivalence between the propositions in this logic, closed linear spaces and the kernel of some matrix $\hat{p}$.

$$[\![p]\!] \Leftrightarrow \overline{[\![p]\!]} \Leftrightarrow \{v | \hat{p}.v = 0\} \,,$$

Given $\hat{p}$, it is possible to express its kernel in FOR as follows

$$\begin{aligned} \mathbb{C} \models \forall v \ \hat{p}.v = 0 \Leftrightarrow C \models & p_{11}.v_1 + p_{12}.v_2 + \ldots + p_{1n}.v_n = 0 \wedge \\ & \ldots \\ & p_{n1}.v_1 + p_{n2}.v_2 + \ldots + p_{nn}.v_n = 0 \,. \end{aligned}$$

The matrix $\hat{p}$ can be determined effectively from the kernel.

- 2022: We finally appear in a picture together!

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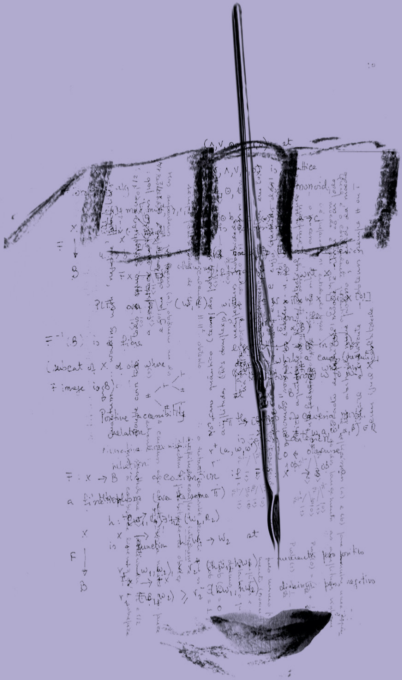
- Multiples future lines of research
  - In the field of logic: The research of axiomatics will be a natural step from here;
- Professor Luís had a vital role in my Ph.d D. so thank you very much Professor!
  - ... but then again according to current life expectancy there are still 14086 chances to invite him for dinner!


F⁻¹(B) is a fibre
(subcat of X of obj whose
F image is B)
Positive compatibility
relation
is a function

Logics and Calculi for all!