A hybrid μ-calculus for monoidal categories

Alejandro Sánchez[1], Luís S. Barbosa[2]

[1]Departamento de Informática
Universidad Nacional de San Luis, San Luis, Argentina

[2]INESC TEC
Universidade do Minho, Braga, Portugal

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Organization

- Motivation
- Background
- The logic
- Results on hybrid μ-calculi
- Ongoing and future work

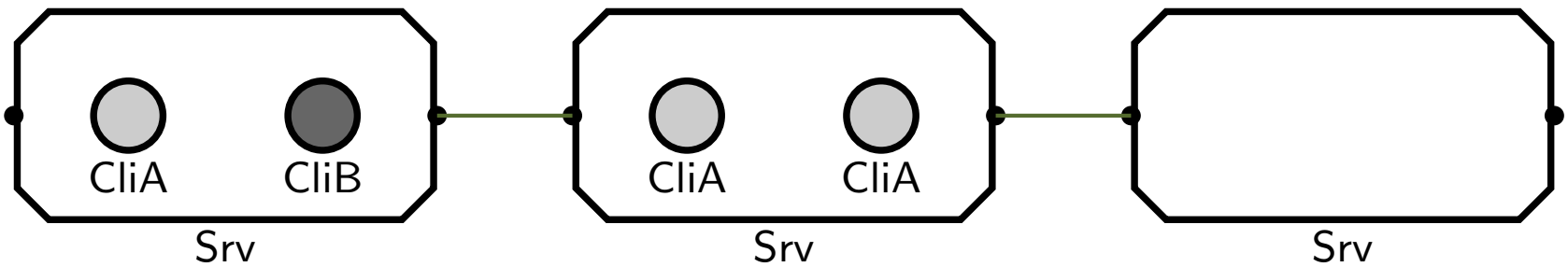
Motivation

- Monoidal categories are pervasive in Computer Science
- They have a graphical notation (Selinger 2011)
- Strict symmetric monoidal categories as models of concurrency
  - Petri nets (Meseguer y Montanari 1990)
  - Bigraphs (Milner 2009)
    - Orthogonality of locality and connectivity
    - CCS, π-calculus, Mobile Ambients, Petri Nets
- We use bigraphs for reasoning about software architectures (and Bigraphical Reactive Systems (Milner 2009) for studying their reconfiguration)

We aim at a general logic for reasoning about bigraphs (and structures in monoidal categories)

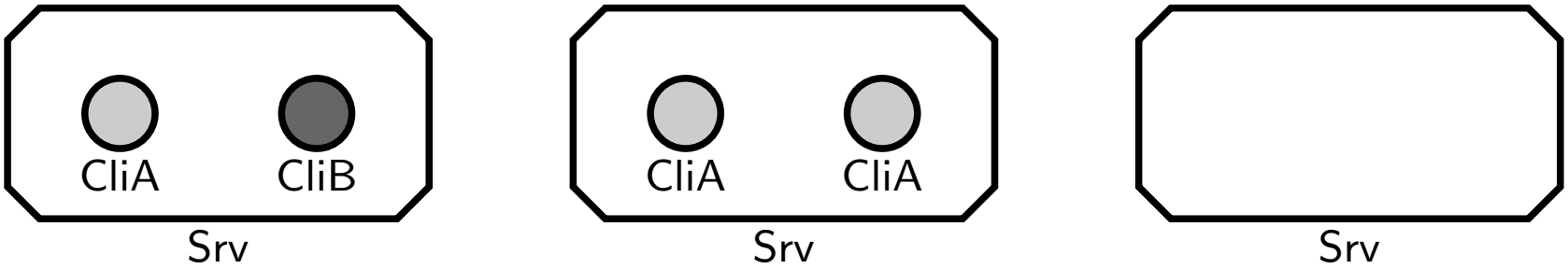
Motivation: Verifying bigraphical structures

A configuration of collaborating services
connected through two ports
and clients (of two kinds) registered to access them



- Only clients CliA are registered in the first service

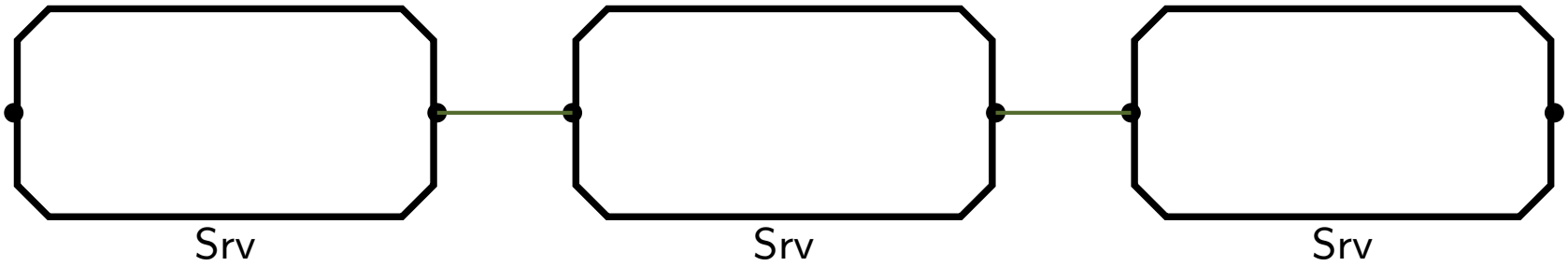
Background: Placegraph



- Parent-child relationship among nodes - Forrest
- Composable if their interfaces - ordinals - agree
- Can be placed one besides the other
- Arrows in a Placegraph Category $\mathrm{Pg}(\mathcal{K})$ (objects are ordinals) for a signature $\mathcal{K}$ ($\{\mathsf{Srv}, \mathsf{CliA}, \mathsf{CliB}\}$ in the example)
- Elementary placegraphs:
  $1 : 0 \to 1$, $\gamma_{1,1} : 2 \to 2$, *join* $: 2 \to 1$, $\mathsf{K} : 1 \to 1$ (K abbr. $\mathsf{K} \circ 1$)

$(\mathsf{Srv} \circ \mathit{join} \circ (\mathsf{CliA} \otimes \mathsf{CliB})) \otimes (\mathsf{Srv} \circ \mathit{join} \circ (\mathsf{CliA} \otimes \mathsf{CliA})) \otimes \mathsf{Srv}$

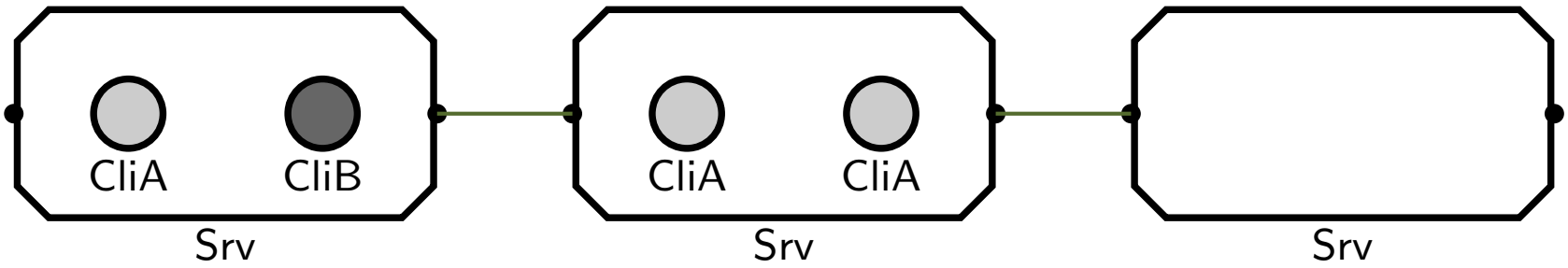
Background: Linkgraphs



- Edges relate nodes, connected to their ports - Hypergraph
- Composable if interfaces - (sets of) names - agree
- Can be placed one besides the other if they have disjoint names
- Arrows in a Linkgraph Category $\mathsf{Lg}(\mathcal{K})$ (objects are names)
- Elementary linkgraphs:
  $y/x : X \to y$, $/x : x \to \emptyset$, $\mathsf{K}_{\vec{x}} : \emptyset \to \{\vec{x}\}$

$$/\{wxyz\} \circ (\mathsf{Srv}_{\overrightarrow{wx}} \otimes \mathsf{Srv}_{\overrightarrow{xy}} \otimes \mathsf{Srv}_{\overrightarrow{yz}})$$

Background: Bigraphs



- Bigraphs combine placegraphs and linkgraphs (same nodes)
- Composable if interfaces - ordinals and names - agree
- Can be placed one besides the other if their linkgraphs allow
- Arrows in a Bigraph Category $\mathsf{Bg}(\mathcal{K})$
  (objects are pairs of ordinals and names)
- Elementary bigraphs: $\mathsf{K}_{\vec{x}} : 1 \to \langle 1, \{\vec{x}\}\rangle$

$/\{wxyz\} \circ$
$((\mathsf{Srv}_{\vec{wx}} \circ \mathit{join} \circ (\mathsf{CliA} \otimes \mathsf{CliB})) \otimes (\mathsf{Srv}_{\vec{xy}} \circ \mathit{join} \circ (\mathsf{CliA} \otimes \mathsf{CliA})) \otimes \mathsf{Srv}_{\vec{yz}})$

Background: Hybrid μ-calculus (Sattler y Vardi 2001)

Given a set $Prop$ of propositional symbols, a set Mod of modal symbols, a set Nom of nominals and a set Var of variables

$$\begin{array}{lcl} \varphi & \ni & \top \mid \bot \mid \varphi \wedge \varphi' \mid \varphi \vee \varphi' \mid p \mid \neg p \mid \\ & & \langle \alpha \rangle \varphi \mid [\alpha]\, \varphi \mid X \mid \mu X.\varphi(X) \mid \nu X.\varphi(X) \\ \alpha & \ni & m \mid m^{\circ} \mid o \end{array}$$

with $p \in Prop \cup Nom$, $m \in Mod$ and $X \in Var$

$$@_p \varphi = [o]\,(\neg p \vee \varphi)$$

$$\mathfrak{M} = (W, \{R_m\}_{m \in Mod}, V)$$

- W is a set of worlds
- $\{R_m\}_{m \in Mod}$ a family of accessibility relations $R_m \subseteq W \times W$
- $V : Prop \cup Nom \to \mathcal{P}(W)$ is a valuation function

$L_{H\mu}(\mathsf{M})$: A hybrid μ-calculus for monoidal categories

Given a monoidal category $\mathsf{M} = (Obj_{\mathsf{M}}, Hom_{\mathsf{M}}, \circ, id, \otimes, \epsilon, N)$ with natural isomorphisms N and a structural congruence $\cong$ induced by its axioms and coherence conditions

the $L_{H\mu}(\mathsf{M})$ is instantiated with:

- Worlds as structural equivalence classes of arrows
- Accessibility relations as binary (de)compositions of arrows
- Mod containing a symbol:
  - $\circ$ for the vertical composition
  - $\otimes$ for the horizontal composition symbol
- $Prop$ including a symbol:
  - id_A for the identity of each object in Obj_{M}
  - for each natural isomorphism in N
  - for each elementary arrow

$L_{H\mu}$(M): Syntax

Given Mod a set of modal symbols, $Prop$ a set of propositional symbols, Nom a set of nominals and Var a set of variables

$$\begin{aligned}\varphi \quad \ni \quad & \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \\ & \langle m \rangle\ \varphi\ \varphi' \mid \\ & [m]\ \varphi\ \varphi' \mid \\ & p \mid \neg p \mid @_p \varphi \mid \\ & X \mid \mu X.\varphi(X) \mid \nu X.\varphi(X)\end{aligned}$$

with $m \in Mod$, $p \in Prop \cup Nom$ and $X \in Var$

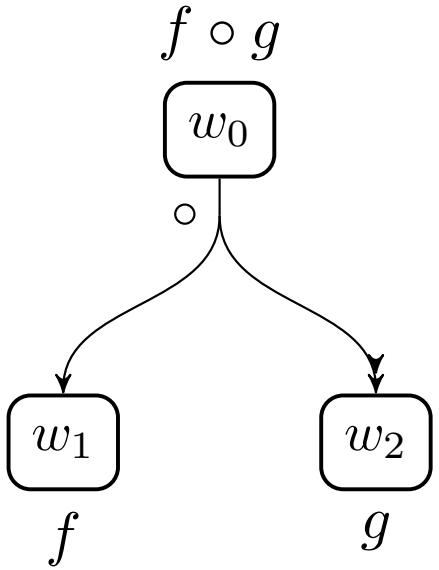
- $\langle m \rangle\ \varphi\ \varphi'$:
  exists a binary decomposition in arrows satisfying φ and φ'
- $[m]\ \varphi\ \varphi'$:
  all binary decompositions yield arrows satisfying φ and φ'

$L_{H\mu}(\mathrm{M})$: Interpretation model

$$\mathfrak{M} = (W, \{R_m\}_{m \in Mod}, V)$$

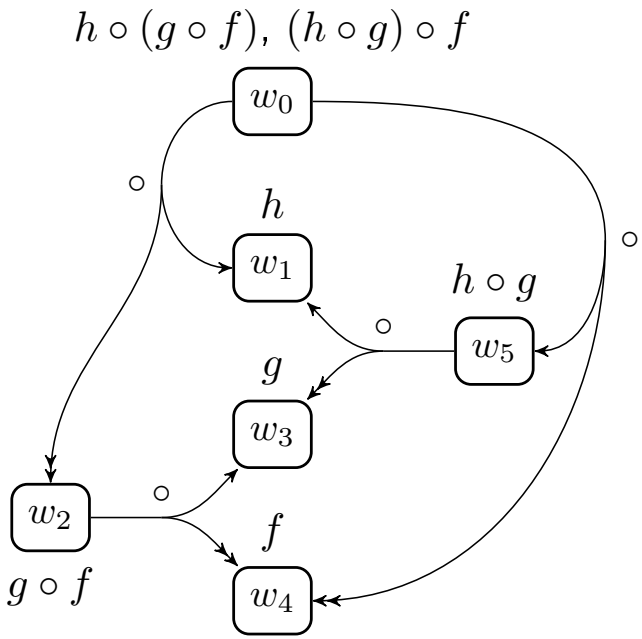
- W: a set of worlds
- $\{R_m\}_{m \in Mod}$:
  a family of accessibility relations $R_m \subseteq W \times (W \times W)$
- $V : Prop \cup Nom \to \mathcal{P}(W)$ is a valuation function

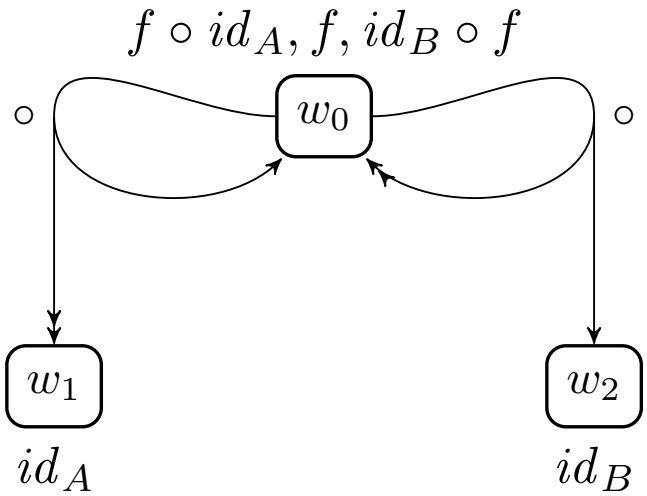
Partial interpretation model
for $f \circ g$



$L_{H\mu}$(M): Interpretation model - Category axioms

- *Associativity*: $h \circ (g \circ f) = (h \circ g) \circ f$
  for all arrows $f : A \to B$, $g : B \to C$ and $h : C \to D$
- *Unit*: $f \circ id_A = f = id_B \circ f$
  for all arrows $f : A \to B$





The interpretation function $[\![\cdot]\!]_{\mathfrak{v}} : L_{H\mu}(\mathsf{M}) \to \mathcal{P}(W)$,
for an environment $\mathfrak{v} : Var \to \mathcal{P}(W)$,
is inductively defined as

$$[\![\top]\!]_{\mathfrak{v}} \triangleq W \tag{1}$$

$$[\![\bot]\!]_{\mathfrak{v}} \triangleq \emptyset \tag{2}$$

$$[\![\varphi \wedge \varphi']\!]_{\mathfrak{v}} \triangleq [\![\varphi]\!]_{\mathfrak{v}} \cap [\![\varphi']\!]_{\mathfrak{v}} \tag{3}$$

$$[\![\varphi \vee \varphi']\!]_{\mathfrak{v}} \triangleq [\![\varphi]\!]_{\mathfrak{v}} \cup [\![\varphi']\!]_{\mathfrak{v}} \tag{4}$$

$$[\![\langle m \rangle\ \varphi\ \varphi']\!]_{\mathfrak{v}} \triangleq \{v \mid \exists w, w' \in W : (v, (w, w')) \in R_m \wedge w \in [\![\varphi]\!]_{\mathfrak{v}} \wedge w' \in [\![\varphi']\!]_{\mathfrak{v}}\} \tag{5}$$

$$[\![\,[m]\ \varphi\ \varphi']\!]_{\mathfrak{v}} \triangleq \{v \mid \forall w, w' \in W : (v, (w, w')) \in R_m \to w \in [\![\varphi]\!]_{\mathfrak{v}} \wedge w' \in [\![\varphi']\!]_{\mathfrak{v}}\} \tag{6}$$

The interpretation function $[\![\cdot]\!]_{\mathfrak{v}} : L_{H\mu}(\mathsf{M}) \to \mathcal{P}(W)$,
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$$[\![\langle m \rangle\ \varphi\ \varphi']\!]_{\mathfrak{v}} \triangleq \{v \mid \exists w, w' \in W : (v, (w, w')) \in R_m \wedge w \in [\![\varphi]\!]_{\mathfrak{v}} \wedge w' \in [\![\varphi']\!]_{\mathfrak{v}}\} \tag{5}$$

$$[\![\,[m]\ \varphi\ \varphi']\!]_{\mathfrak{v}} \triangleq \{v \mid \forall w, w' \in W : (v, (w, w')) \in R_m \to w \in [\![\varphi]\!]_{\mathfrak{v}} \wedge w' \in [\![\varphi']\!]_{\mathfrak{v}}\} \tag{6}$$

$$[m]\ \varphi\ \varphi' \neq \neg\langle m \rangle\ \neg\varphi\ \neg\varphi'$$

$$[\![p]\!]_{\mathfrak{v}} \triangleq V(p) \tag{7}$$

$$[\![\neg p]\!]_{\mathfrak{v}} \triangleq W \setminus V(p) \tag{8}$$

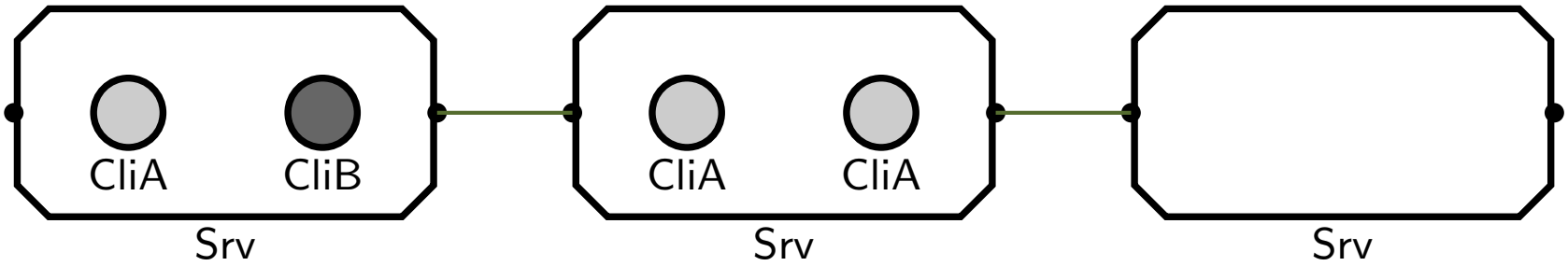
$$[\![@_p\varphi]\!]_{\mathfrak{v}} \triangleq \begin{cases} W & \text{if } V(p) \subseteq [\![\varphi]\!]_{\mathfrak{v}} \\ \emptyset & otherwise \end{cases} \tag{9}$$

$$[\![X]\!]_{\mathfrak{v}} \triangleq \mathfrak{v}(X) \tag{10}$$

$$[\![\mu X.\varphi(X)]\!]_{\mathfrak{v}} \triangleq \bigcap \{W' \subseteq W \mid W' \supseteq [\![\varphi(X)]\!]_{\mathfrak{v}[X \mapsto W']}\} \tag{11}$$

$$[\![\nu X.\varphi(X)]\!]_{\mathfrak{v}} \triangleq \bigcup \{W' \subseteq W \mid W' \subseteq [\![\varphi(X)]\!]_{\mathfrak{v}[X \mapsto W']}\} \tag{12}$$

$L_{H\mu}$(M): Example property



$/\{xyz\}\circ$
$((\mathsf{Srv}_{\overrightarrow{xy}} \circ \mathit{join} \circ (\mathsf{CliA} \otimes \mathsf{CliB})) \otimes (\mathsf{Srv}_{\overrightarrow{yz}} \circ \mathit{join} \circ (\mathsf{CliA} \otimes \mathsf{CliA})) \otimes \mathsf{Srv}_{\overrightarrow{zy}})$

- Only clients CliA are registered in the first service

$$@_i\langle\circ\rangle\ \mathsf{Srv}\ \mu X.(1 \vee \mathsf{CliA} \vee\ [\circ]\ \mathit{join}\ [\otimes]\ X\ X)$$

Results on hybrid μ-calculi

- Hybrid μ-calculus with global and converse modalities (Sattler y Vardi 2001)
  - Deciding satisfiability is ExpTime
- Fully hybrid μ-calculus (Kernberger y Lange 2017)
  - Includes a binder operator
  - Model checking is ExpTime
  - Invariant under hybrid k-bisimulations
- Hybrid μ-calculus
  - The finite model property (Tamura 2013)
  - Sound and complete proof system (Enqvist 2020)

Ongoing and future work

- Verify bigraphical structures (modelling software architectures)
- Reason about structures in other monoidal categories
  - Petri net categories (Meseguer y Montanari 1990)
  - Graphical notation (Selinger 2011)
- Study properties of the logic
  - Does it has a form bisimulation invariance?
  - Is still ExpTime deciding satisfiability? (Sattler y Vardi 2001)
  - Does it has the finite model property? (Tamura 2013)
  - Does it has a sound and complete proof system? (Enqvist 2020)

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