Phenomenology of Computation

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Historical Perspective

<u>Moggi 1988</u>: how to incorporate "effects" into denotational semantics?

Answer: Monads!

<u>Plotkin, Power (and many others)</u>: computational effects algebraically.

Monads are given by operations and equations.

<u>The point of view of Category Theory</u>: equational theories are Lawvere theories.

Monads $\Longrightarrow$ Lawvere theories

Historical Perspective

<u>Giry monad (1981)</u>: on measure spaces and also on Polish spaces.

A monad of great interest: <u>Lawvere</u> (1964) The category of probabilistic mappings (unpublished).

Metrics ideas present in semantics from the start: Jaco de Bakker's school

Historical Perspective

<u>Giry monad (1981)</u>: on measure spaces and also on Polish spaces.

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Can we integrate metric and topological concepts in our semantics?

Metrics ideas present in semantics from the start: Jaco de Bakker's school

Part 1: Quantitative Algebraic Reasoning

(LICS'16)

<u>Our goal</u>: develop the theory of effects in a metric settings.

- algebras shall come with a metric structure
- <u>quantitative equational theories</u> will define monads on $\mathbb{M}\mathrm{et}$

An Example

Barycentric Algebras (M.H. Stone 1949)

Signature: $\mathcal{B} = \{ +_\varepsilon \ / \ \varepsilon \in [0,1] \}$

Axioms:

(B_1): $\vdash t +_1 t' = t$

(B_2): $\vdash t +_\varepsilon t = t$

(SC): $\vdash t +_\varepsilon t' = t' +_{1-\varepsilon} t$

(SA): $\vdash (t +_\varepsilon t') +_{\varepsilon'} t'' = t +_{\varepsilon\varepsilon'} \left(t' +_{\frac{\varepsilon' - \varepsilon\varepsilon'}{1-\varepsilon\varepsilon'}} t'' \right)$

Models:

$\mathbb{R}$ with Euclidean distance

$$a +_\varepsilon b \stackrel{df}{=} \varepsilon a + (1-\varepsilon) b$$

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Models:

(X, Σ) – measurable space, $\Delta(X, \Sigma)$ – prob. distrib.

$$\mu +_\varepsilon \nu \overset{df}{=} \varepsilon \mu + (1-\varepsilon)\nu$$

Equational Theory

Given an algebraical similarity type Ω, an equational theory defines a congruence relation on terms

<u>Meta-axioms</u>:

(Refl): $\emptyset \vdash t = t$

(Symm): $s = t \vdash t = s$

(Trans): $\{s = t, t = u\} \vdash s = u$

(Cong): $\{s_i = t_i \mid i = 1, n\} \vdash f(s_1 .. s_n) = f(t_1 .. t_n)$, $f : n \in \Omega$

. . . . . .

Quantitative Equational Theory

congruence on terms $\Longrightarrow$ pseudometric (uniformity) on terms

equations: $s = t$ $\Longrightarrow$ quantitative equations $s =_\varepsilon t$

interpreted as "s is within ε of t"

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<u>Meta-axioms</u>

(Trans): $\{s = t,\ t = u\} \vdash s = u$

$\Uparrow$

(Triang): $\{s =_\varepsilon t,\ t =_\delta u\} \vdash s =_{\varepsilon+\delta} t$

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(Cong): $\{s_i = t_i \,/\, i = 1,n\} \vdash f(s_1 .. s_n) = f(t_1 .. t_n)$, $f : n \in \Omega$

$\Uparrow$

(NExp): $\{s_i =_\varepsilon t_i \,/\, i = 1,n\} \vdash f(s_1 .. s_n) =_\varepsilon f(t_1 .. t_n)$, $f : n \in \Omega$

Quantitative Equational Theories

Signature Ω, variables X $\longrightarrow$ terms $\hat{\Omega} X$

Quantitative equations on terms

$$s =_\varepsilon t \qquad \varepsilon \in \mathbb{R}_+$$

Quantitative judgements

$$\{ s_i =_{\varepsilon_i} t_i \mid i \in I \} \vdash s =_\varepsilon t$$

Quantitative Equational Theory: a set of judgements closed under the quantitative deducibility meta-axioms

Quantitative Algebra

Given a signature Ω, $\mathcal{A} = (A, \Omega, d)$

- (A, Ω) is an Ω-algebra
- (A, d) is a metric space
- all functions in Ω are non-expansive

Morphisms of QAs : nonexpansive Ω-algebra homomorphisms

$$f : (A_1, \Omega, d_1) \longrightarrow (A_2, \Omega, d_2)$$

- $f : (A_1, \Omega) \longrightarrow (A_2, \Omega)$ – Ω-homomorphism
- $f : (A_1, d_1) \longrightarrow (A_2, d_2)$ – nonexpansive map

Algebraic Semantics

Given a QA $\mathcal{A} = (A, \Omega, d)$

$\mathcal{A}$ satisfies $\{ s_i =_{\varepsilon_i} t_i \,/\, i \in I \} \vdash s =_\varepsilon t$

$\{ s_i =_{\varepsilon_i} t_i \,/\, i \in I \} \models_{\mathcal{A}} s =_\varepsilon t$

$\forall$ assignment $\alpha : \hat{\Omega} X \longrightarrow \mathcal{A}$ — Ω-homomorphism

$[\, d(\alpha(s_i), \alpha(t_i)) \leq \varepsilon_i \;, i \in I \,]$ implies $d(\alpha(s), \alpha(t)) \leq \varepsilon$

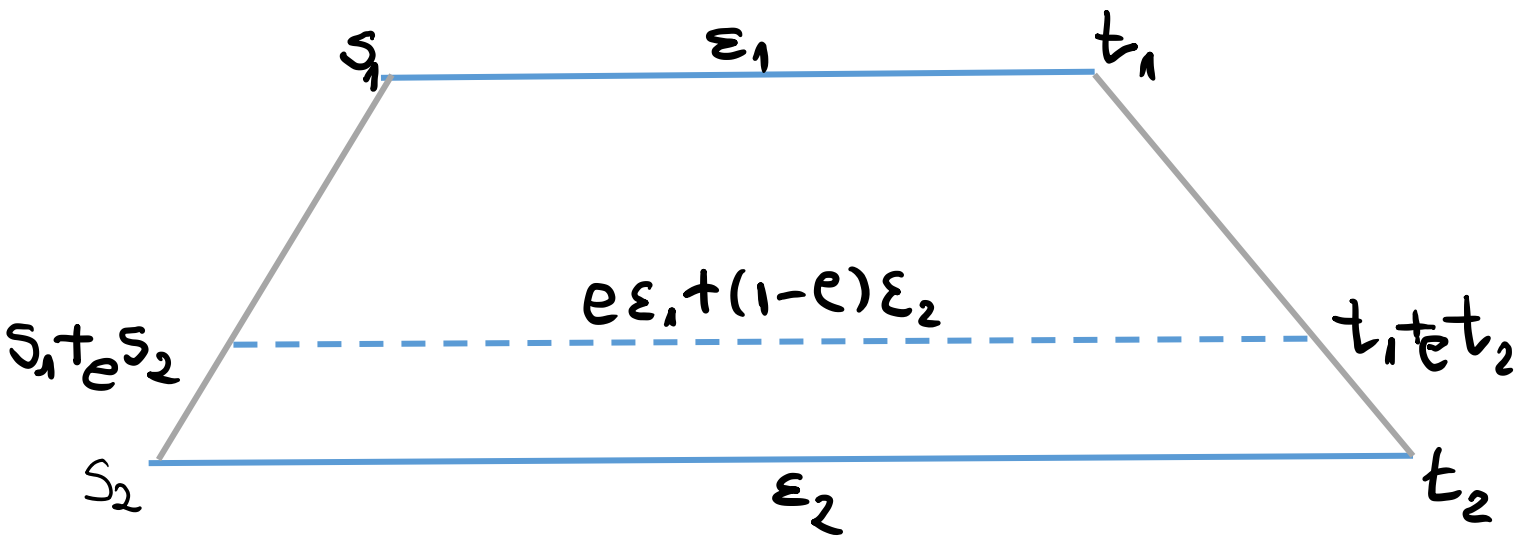
Example: Interpolative Barycentric Algebras

Signature: $\mathcal{B} = \{ +_e \mid e \in [0,1] \}$

Axioms: (B_1), (B_2), (SC), (SA)

$$(E): \; s_1 =_{\varepsilon_1} t_1 \,,\; s_2 =_{\varepsilon_2} t_2 \vdash s_1 +_e s_2 =_{\delta} t_1 +_e t_2$$

$$\delta = e\varepsilon_1 + (1-e)\varepsilon_2$$



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<u>Models:</u>

$\mathbb{R}$ with Euclidean distance

$$a +_\varepsilon b \overset{df}{=} \varepsilon a + (1-\varepsilon) b$$

(X, Σ) – measurable space $\quad \Delta(X, \Sigma)$ – prob. distrib

$$\mu +_\varepsilon \nu \overset{df}{=} \varepsilon \mu + (1-\varepsilon)\nu$$

Completeness

Given a theory $\mathcal{U}$ over $\hat{\Omega}X$

$\mathcal{M}(\Omega, \mathcal{U})$ the set of all QAs satisfying $\mathcal{U}$.

Theorem [Completeness, LICS'16]:

For any judgement $\Gamma \vdash \varphi$ on $\hat{\Omega}X$,

$\Gamma \vdash \varphi \in \mathcal{U}$ iff $[\forall \mathcal{A} \in \mathcal{M}(\Omega, \mathcal{U}),\ \Gamma \models_{\mathcal{A}} \varphi]$

Term Algebra

Given $\mathcal{U}$ over $\hat{\Omega}X$

$$d^{\mathcal{U}} : \hat{\Omega}X \times \hat{\Omega}X \to \overline{\mathbb{R}}_+$$

$$d^{\mathcal{U}}(s,t) = \inf\{\varepsilon \mid \vdash s =_\varepsilon t \in \mathcal{U}\}$$

$$\Omega[X] = (\hat{\Omega}X, \Omega, d^{\mathcal{U}})$$ – after quotieting by $\ker(d^{\mathcal{U}})$

Theorem [Quantitative term algebra]:

$$\Omega[X] \in \mathcal{M}(\Omega, \mathcal{U})$$

Generated Model

Given $\mathcal{U}$ over $\hat{\Omega}X$

Given (M, d_M) - metric space

$$d_M^{\mathcal{U}} : \hat{\Omega}M \times \hat{\Omega}M \longrightarrow \overline{\mathbb{R}}_+$$

$$d_M^{\mathcal{U}}(m,n) = \inf\{\varepsilon \,/\, \exists\ s =_\varepsilon t \in \mathcal{U} \text{ and } \exists\, \alpha \text{ s.t. } \alpha(m) = s,\ \alpha(n) = t\}$$

the final pseudometric for the cone of assignments to $\hat{\Omega}M$

$$\Omega[M] = (\hat{\Omega}M, \Omega, d_M^{\mathcal{U}})$$

Theorem [Freely generated model]:

$$\Omega[M] \in \mathcal{M}(\Omega, \mathcal{U})$$

Universality

Theorem: $\Omega[M]$ has the universal mapping property for (M,d) to the forgetfull functor $U : \mathbb{K}(\Omega,\mathcal{U}) \longrightarrow \mathbb{M}\mathrm{et}$

$$\begin{tikzcd}
\mathbb{M}\mathrm{et} & \\
(M,d) \arrow[r, "\eta_M"] \arrow[dr, "\alpha"'] & U(\Omega[M]) \arrow[d, "Uh"] \\
& U\mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K}(\Omega,\mathcal{U}) & \\
\Omega[M] \arrow[d, dashed, "h"] & \\
\mathcal{A} &
\end{tikzcd}$$

Hence, $\Omega[-]$ defines a monad on $\mathbb{M}\mathrm{et}$

A landscape

$$\begin{tikzcd}
\text{Monads on } \mathbb{M}\mathrm{et}\ \Omega^{\mathcal{U}} \arrow[dr] & \\
& \Omega^{\mathcal{U}}\text{-EM} \cong \mathcal{M}\mathrm{o}(\Omega, \mathcal{U}) \\
\text{Quantitative Equational Theories}\ \mathcal{U} \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Example: Interpolative Barycentric Algebras

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Axioms:

$$\left.\begin{array}{ll}
(B_1): & \vdash t +_1 t' =_0 t \\
(B_2): & \vdash t +_\varepsilon t =_0 t \\
(SC): & \vdash t +_\varepsilon t' =_0 t' +_{1-\varepsilon} t \\
(SA): & \vdash (t +_\varepsilon t') +_{\varepsilon'} t'' =_0 t -_{\varepsilon\varepsilon'} \left(t' +_{\frac{\varepsilon' - \varepsilon\varepsilon'}{1-\varepsilon\varepsilon'}} t'' \right) \\
(E): & s_1 =_{\varepsilon_1} t_1 ,\ s_2 =_{\varepsilon_2} t_2 \vdash s_1 +_e s_2 =_{e\varepsilon_1 + (1-e)\varepsilon_2} t_1 +_e t_2
\end{array}\right\} \mathcal{E}$$

We can construct

$$\mathcal{B}[X] = (\hat{\mathcal{B}}X, \mathcal{B}, d^{\mathcal{E}})$$

Example: Interpolative Barycentric Algebras

(M, δ) metric space and $\Pi(M)$ – probab. distr. with finite support

$$K : \Pi(M) \times \Pi(M) \longrightarrow \overline{\mathbb{R}}_+$$

– Kantorovich distance

$$K(\mu, \nu) = \inf \left\{ \int \delta \, d\omega \;/\; \omega \in \mathcal{C}(\mu, \nu) \right\}$$

where $\mathcal{C}(\mu, \nu)$ is the set of couplings (prab. distrib. on $M \times M$) with marginals μ and ν

Example: Interpolative Barycentric Algebras

(M, δ) metric space and $\Pi(M)$ - prob. distr. with finite support and K Kantorovich distance on $\Pi(M)$

Theorem:

$$\mathcal{B}[M] = (\hat{\mathcal{B}}M, \mathcal{B}, \delta^{\mathcal{E}}) \overset{\text{iso}}{\cong} (\Pi(M), \mathcal{B}, K)$$

Hence, $\mathcal{E}$ axiomatizes Kantorovich distance for finitely supported distributions!

Example: Interpolative Barycentric Algebras

(M, δ) - complete separable metric space

$\Delta(M)$ - Borel prob. measures on (M, δ), K Kantorovich on $\Delta(M)$

Let $\overline{B[M]}$ be the quant. algebra obtained by completing the metric space $(\hat{B}M, \delta^{\mathcal{E}})$ and defining operators to the limit

Theorem: $\overline{B[M]} \overset{\text{iso}}{\cong} (\Delta(M), \mathcal{B}, K)$

Hence, $\mathcal{E}$ axiomatizes the Kantorovich metric!

Example: Quantitative Semilattices with Zero

Signature: $\Sigma = \{+, 0\}$

Axioms:

$$\left.\begin{array}{ll}
(S_0) & \vdash t + 0 =_0 t \\
(S_1) & \vdash t + t =_0 t \\
(S_2) & \vdash t + t' =_0 t' + t \\
(S_3) & \vdash (t + t') + t'' =_0 t + (t' + t'') \\
(S_4) & s_1 =_{\varepsilon_1} t_1 \,,\, s_2 =_{\varepsilon_2} t_2 \;\vdash\; s_1 + s_2 =_{\max\{\varepsilon_1, \varepsilon_2\}} t_1 + t_2
\end{array}\right\} \mathcal{S}$$

Quantitative Semilattices with Zero: Finitary Case

Given a metric space (M, δ)

Let $\mathbb{F}(M)$ be the set of all finite subsets of M

Let $A + B = A \cup B$ and $0 = \emptyset$

Let H^{δ} be the Hausdorff distance on $\mathbb{F}(M)$ induced by δ.

Theorem:

$$\Sigma[M] = (\hat{\Sigma} M, \Sigma, \delta^{\mathcal{S}}) \overset{\text{iso}}{\cong} (\mathbb{F}(M), \Sigma, H^{\delta})$$

Hence, $\mathcal{S}$ axiomatizes Hausdorff on finite sets!

Quantitative Semilattices with Zero: Continuous Case

Let (M,d) – complete separable metric space

$\mathbb{G}(M)$ – the set of all closed subsets of M

H^{δ} – Hausdorff distance induced by δ on $\mathbb{G}(M)$

Let $\overline{\Sigma[M]}$ be the quant. alg. induced by the completion of $(\hat{\Sigma} M, \delta^{s})$ and defining the operators to the limit

Theorem: $\overline{\Sigma[M]} \overset{iso}{\cong} (\mathbb{G}(M), \Sigma, H^{\delta})$

Hence, $\mathcal{S}$ axiomatizes the Hausdorff distance!

Other results

- Generalizations of the Birkhoff's varieties and quasi-varieties theorems (LICS'17)
- Disjoint unions of quantitative theories and their corresponding monads (LICS'18)
- Tensorial products of quantitative theories and their corresponding monads (CALCO'21)
- Many relevant examples for applications

Part 2: Quantitative Iteration Theories

(LICS'21)

Can we extend quantitative equational reasoning with fixed points?

(N.B.: the fixed points are not algebraic)

The classic iteration theories

- Multisorted parametrized fixed-point extension of an algebraic theory for Ω and $t \in \hat{\Omega} X \implies \mu x.t$ – least fixed point
- A theory is a (typed) congruence on terms under certain restrictions (variable binding, α-conversion, etc)
- Traditionally are interpreted on "domain-like" categories and exploit the Kleene's or Knaster–Tarski fixed point theorems [Bloom–Esik, Hasegawa]
- Plotkin and Simpson propose axiomatic definitions for Conway and for iteration theories that can, in principle, be interpreted in different kind of categories.

Banach fixed point theorem

Let (X, d) - complete metric space

$T : X \longrightarrow X$ - contractive map: $\exists\, c \in (0,1)$ s.t. $d(Tx, Ty) \leq c \cdot d(x, y)$

Then, T admits a <u>unique</u> fixed point $\mu x.\, Tx$

Moreover, $\forall y \in X$ $\lim\limits_{n \to \infty} T^n y = \mu x.\, Tx$ where $T^n y = T(T^{n-1} y)$

Also, if we know $d(y, Ty) \leq \varepsilon$, we can describe the geometry of the convergence of $(T^n y)_{n \geq 0}$

Banach patterns

Let $f : (A,d)^n \longrightarrow (A,d)$

$\theta \subseteq \{ \bar{\alpha} = \langle \alpha_1 \dots \alpha_n \rangle \in [0,1]^n \,/\, \sum \alpha_i \leq 1 \}$

θ is a <u>Banach pattern for f</u> if:

$$\forall \bar{a}, \bar{b} \in A^n, \quad d(f(\bar{a}), f(\bar{b})) \leq \max_{\bar{\alpha} \in \theta} \sum_{i=1}^{n} \alpha_i \, d(a_i, b_i)$$

<u>Examples:</u> $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\quad f(x,y) = x +_\varepsilon y$

then:

$d(f(x,y), f(x',y')) \leq d(x,x') \implies \{\langle 1,0 \rangle\}$ is a BP for f

$\leq d(y,y') \implies \{\langle 0,1 \rangle\}$ is a BP for f

$\leq \max\{d(x,x'), d(y,y')\} \implies \{\langle 0,1 \rangle, \langle 1,0 \rangle\}$ is a BP for f

$\leq \varepsilon \cdot d(x,x') + (1-\varepsilon) d(y,y') \implies \{\langle \varepsilon, 1-\varepsilon \rangle\}$ is a BP for f

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<u>Examples:</u> $f : A^n \longrightarrow A$

- f is <u>nonexpansive in its i-th variable</u> iff $\{\langle 0 \dots \overset{i^{th}\text{position}}{1} \dots 0 \rangle\}$ is a BP for f
- f is <u>nonexpansive</u> iff $\{\langle 1, 0 \dots 0 \rangle, \langle 0, 1, 0 \dots 0 \rangle, \dots \langle 0 \dots 0, 1 \rangle\}$ is a BP for f
- f is <u>c-Lipschitz</u> in its i-th variable iff $\{\langle 0 \dots \overset{i\text{-th position}}{c} \dots 0 \rangle\}$ is a BP for f

Banach Theory

Banach signature: $\Omega \ni f : n : \theta$, θ is a BP for f

(Congruence): $\{x_i = y_i \mid i \in I\} \vdash f(\bar{x}) = f(\bar{y})$ — universal algebras

$$\Downarrow$$

(Nexp): $\{x_i =_\varepsilon y_i \mid i \in I\} \vdash f(\bar{x}) =_\varepsilon f(\bar{y})$ — quantitative algebras

$$\Downarrow$$

(Banach): $\{x_i =_{\varepsilon_i} y_i \mid i \in I\} \vdash f(\bar{x}) =_{\max_{\bar{\alpha} \in \theta} \sum \alpha_i \varepsilon_i} f(\bar{y})$ — Banach theories

The Clone of Banach patterns

BPs can be properly defined for a clone of functions $\begin{cases} \text{- compositions} \\ \text{- abstractions} \\ \text{- projections} \end{cases}$

and for fixed points

Given $f : n : \theta \triangleright i$ – meaning $\forall \bar{\alpha} \in \theta,\ \alpha_i < 1$

$$\Downarrow$$

$\mu i . f : n-1 : \mu i . \theta$ where for $\bar{\alpha} \in \theta$,

$$\mu i . \bar{\alpha} = \left\langle \frac{\alpha_1}{1-\alpha_i} \cdots \frac{\alpha_{i-1}}{1-\alpha_i}\ \frac{\alpha_{i+1}}{1-\alpha_i} \cdots \frac{\alpha_n}{1-\alpha_i} \right\rangle$$

and

$$\mu i . \theta = \{ \mu i . \bar{\alpha} \ / \ \bar{\alpha} \in \theta \}$$

Fixed-point extensions of Banach theories

Let $\mathcal{U}$ be a Banach theory over $\hat{\Omega} X$

Let $\widehat{\Omega^{\mu} X}$ – the closure of $\hat{\Omega} X$ under fixed-point constructions

Let $\mathcal{U}^{\mu}$ – the smallest quantitative equational theory over $\widehat{\Omega^{\mu} X}$

– that extends $\mathcal{U}$: $\mathcal{U} \subseteq \mathcal{U}^{\mu}$

– closed under (Approx)

$$(\text{Approx}): \quad \frac{t : n : \theta \triangleright i}{u =_{\varepsilon} t((u)) \vdash u =_{\frac{\varepsilon}{1-a}} \mu x.\, t((x))} \quad \text{where } a = \max\{\alpha_i \mid \vec{\alpha} \in \theta\}$$

[$t((x))$ is a shortcut for $t(x_1 \ldots x \ldots x_n)$ helping us to "focus" on variable x while treating the others as parameters]

Semantics of fixed-point terms

Let $\mathbb{C}(\Omega)$ – the Ω-quantitative algebras <u>over complete metric spaces</u>

$\mathbb{C}(\mathcal{U})$ – the models of $\mathcal{U}$ in $\mathbb{C}(\Omega)$

For $\mathcal{A} = (A, \Omega, d) \in \mathbb{C}(\mathcal{U})$ and $t : n : \theta \triangleright i \in \widehat{\Omega^{\mu} X}$

$t^{\mathcal{A}} : A^n \to A$ is contractive in its i-th variable

$\Rightarrow \left(t^{k}((a))\right)_{k \geq 0}$ is a Cauchy sequences having the limit $[t^{\mathcal{A}}]^{*}_{i} : A^{n-1} \to A$

Define

$$[\mu x. t((x))]^{\mathcal{A}} \stackrel{\text{def}}{=} [t^{\mathcal{A}}]^{*}_{i}$$

Completeness

Let $\mathcal{U}$ be a Banach theory on $\hat{\Omega} X$

Theorem:

(i) For any $\mathcal{A} \in \mathbb{C}(\mathcal{U})$,

$$\mathcal{A} \models \mathcal{U} \quad \text{iff} \quad \mathcal{A} \models \mathcal{U}^{\mu}$$

(ii) For any fixed-point judgement $\Gamma \vdash \varphi$ in $\widehat{\Omega^{\mu} X}$,

$$\Gamma \models_{\mathbb{C}(\mathcal{U})} \varphi \quad \text{iff} \quad \Gamma \vdash \varphi \in \mathcal{U}^{\mu}$$

Banca theorem

Theorem: Let $\mathcal{U}$ be a Banach theory on $\hat{\Omega}X$, $f : n : \theta \triangleright i \in \widehat{\Omega^{\mu}X}$.

For any $s \in \widehat{\Omega^{\mu}X}$,

(i) $\vdash \mu x.f((x)) =_0 f((\mu x.f((x)))) \in \mathcal{U}^{\mu}$

(ii) $s =_0 f((s)) \vdash s =_0 \mu x.f((x)) \in \mathcal{U}^{\mu}$

(iii) $\forall \varepsilon > 0 \ \exists k \ \forall n > k,$

$$\vdash f^{k}((s)) =_{\varepsilon} f^{n}((s)) \in \mathcal{U}^{\mu}$$

(iv) $\forall \varepsilon > 0 \ \exists k \ \forall n > k$

$$\vdash f^{n}((s)) =_{\varepsilon} \mu x.f((x)) \in \mathcal{U}^{\mu}$$

Conway theories

A fixed point theory is a Conway theory if it satisfies the dinaturality and the diagonal properties [Plotkin, Simpson LICS 2000]

Let $\mathcal{U}$ be a Banach theory

Theorem [Quantitative Dinaturality]: If $f : n : \theta \triangleright i$, $g : m : \xi \triangleright j \in \widehat{\Omega^{\mu} X}$,

$$\vdash \mu y.f((g((y)))) =_0 f((\mu x.g((f((x))))))) \in \mathcal{U}^{\mu}$$

Theorem [Quantitative Diagonal Property]: If $f : n : \theta \in \widehat{\Omega^{\mu} X}$,

$$\vdash \mu x.f((x,x)) =_0 \mu x.\mu y.f((x,y)) \in \mathcal{U}^{\mu}$$

Hence, the quantitative fixed-point theories are Conway theories!

Iteration Theories

A Conway theory is an iteration theory if it satisfies the amalgamation property [Plotkin-Simpson LICS'2000]

Let $\mathcal{U}$ be a Banach theory

Theorem [Quantitative Amalgamation]: If $f_1 : n : \theta^1, \ldots, f_n : n : \theta^n \in \widehat{\Omega^\mu X}$ s.t.

$\forall \bar{\alpha} \in \theta^i, \forall \bar{\beta} \in \theta^j, \ \sum \alpha_k = \sum \beta_k = a < 1.$

Let $g : 1 : \{\langle a \rangle\} \in \widehat{\Omega^\mu X}$ s.t.

for all $i \leq n$ $\quad \vdash f_i(x \ldots x) =_0 g(x) \in \mathcal{U}^\mu$.

If $\exists s_1, \ldots s_n \in \widehat{\Omega^\mu X}$ s.t.

$$\begin{cases} s_1 =_0 f_1(s_1 \ldots s_n) \in \mathcal{U}^\mu \\ s_2 =_0 f_2(s_1 \ldots s_n) \in \mathcal{U}^\mu \\ \ldots \\ s_n =_0 f_n(s_1 \ldots s_n) \in \mathcal{U}^\mu \end{cases}$$

then, for all $i \geq n$,

$\vdash s_i =_0 \mu x. g(x) \in \mathcal{U}^\mu$

Conclusions

Quantitative algebras represent generalizations of universal algebras, where the metric structure of a model interacts properly with the algebraic structure

The results demonstrate that most of the classic constructions and results from universal algebras can be properly generalized

Unlike the classic algebras only capabale of axiomatizing congruences (which can be seen as degenerated metric spaces), the quantitative algebras can axiomatize proper metric spaces.

Being the central role of algebraic reasoning in semantics, the quantitative algebras put the classic computational paradigms in a new light, alowing concepts of approximations to be properly introduced and manipulated.