PNmatrices at work

PNmatrix = Partial non-deterministic matrix

$\mathbb{B} = \langle\{0,1\},\{1\},\cdot_{\mathbb{B}}\rangle$

$\to_{\mathbb{B}}$	0	1
0	1	1
1	0	1

impose $p \to (\neg p \to \neg q)$ $\longrightarrow$

$\mathbb{B}_{\mathsf{Ax}} = \langle\{00,01,10,11\},\{10,11\},\cdot_{\mathbb{B}_{\mathsf{Ax}}}\rangle$

$\to_{\mathbb{B}_{\mathsf{Ax}}}$	00	01	10	11
00	10	10	10	$\emptyset$
01	10	10, 11	10	11
10	00, 01	00, 01	10	$\emptyset$
11	$\emptyset$	01	$\emptyset$	11

	$\neg_{\mathbb{B}_{\mathsf{Ax}}}$
00	00, 01
01	10, 11
10	00, 01
11	11

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Plan: PNmatrices and logics

PNmatrix semantics

Semantical units: PNmatrices $\top$ Logics: consequence relations

'Lindenbaum PNmatrix'

PNmatrices generalize logical matrices by enriching them with partiality and non-determinism.

Good for compositionality results!

We are after correspondence between operations on logics and operations on PNmatrices.

Basic concepts

signatures

$\boldsymbol{\Sigma}$: $\mathbb{N}_0$-indexed set of connectives

$\boldsymbol{\Sigma_1 \cap \Sigma_2 = \{\Sigma_1^{(n)} \cap {\Sigma_2}^{(n)}\}_{n\in\mathbb{N}_0}}$

$\boldsymbol{\Sigma_1 \cup \Sigma_2 = \{\Sigma_1^{(n)} \cup {\Sigma_2}^{(n)}\}_{n\in\mathbb{N}_0}}$

$\boldsymbol{\Sigma_1 \setminus \Sigma_2 = \{\Sigma_1^{(n)} \setminus {\Sigma_2}^{(n)}\}_{n\in\mathbb{N}_0}}$

Propositional languages

$\boldsymbol{L = L_\Sigma(P)}$ given by $\boldsymbol{\psi ::= P \mid ©(\psi, \ldots, \psi)}$ for $\boldsymbol{© \in \Sigma}$

substitutions

$\boldsymbol{\sigma : P \to L}$, $\boldsymbol{\varphi(\vec{\psi}) = \varphi(\vec{p})^\sigma}$ when $\boldsymbol{\sigma(\vec{p}) = \vec{\psi}}$

single-conclusion rules

set $\times$ fmla

$\dfrac{\boldsymbol{\Gamma}}{\boldsymbol{\varphi}}$ with $\boldsymbol{\Gamma, \{\varphi\} \subseteq L}$

Examples: $\dfrac{}{\boldsymbol{p\to(q\to p)}}$, $\dfrac{\boldsymbol{p\,,\, p\to q}}{\boldsymbol{q}}$

multiple-conclusion rules

set $\times$ set

$\dfrac{\boldsymbol{\Gamma}}{\boldsymbol{\Delta}}$ with $\boldsymbol{\Gamma, \Delta \subseteq L}$

Examples: $\dfrac{}{\boldsymbol{p\to(q\to p)}}$, $\dfrac{\boldsymbol{p\,,\, p\to q}}{\boldsymbol{q}}$, $\dfrac{\boldsymbol{p\lor q}}{\boldsymbol{p\,,\, q}}$

Single- and multiple-conclusion logics

A Scottian consequence relation (**set** $\times$ **set**-cr) is a $\rhd \subseteq \wp(L) \times \wp(L)$ satisfying:

$\Gamma \rhd \Delta$ if $\Gamma \cap \Delta \neq \emptyset$ (*overlap*)

$\Gamma \cup \Gamma' \rhd \Delta \cup \Delta'$ if $\Gamma \rhd \Delta$ (*dilution*)

$\Gamma \rhd \Delta$ if $\Gamma \cup \Omega \rhd \overline{\Omega} \cup \Delta'$ for every partition $\langle \Omega, \overline{\Omega} \rangle$ of some $\Theta \subseteq L$ (*cut for sets*)

$\Gamma^\sigma \rhd \Delta^\sigma$ for any substitution $\sigma : P \to L$ if $\Gamma \rhd \Delta$ (*substitution invariance*)

Given a **set** $\times$ **set**-cr $\rhd$, its single conclusion fragment $\vdash_\rhd = \rhd \cap (\wp(L) \times L)$ is a Tarskian consequence relation (**set** $\times$ **fmla**-cr) satisfying:

$\Gamma \vdash \varphi$ if $\varphi \in \Gamma$ (*reflexivity*),

$\Gamma \cup \Gamma' \vdash \varphi$ if $\Gamma \vdash \varphi$ (*monotonicity*),

$\Gamma \vdash \varphi$ if $\Delta \vdash \varphi$ and $\Gamma \vdash \psi$ for every $\psi \in \Delta$ (*transitivity*)

$\Gamma^\sigma \vdash \varphi^\sigma$ for any substitution $\sigma : P \to L$ if $\Gamma \vdash \varphi$ (*substitution invariance*)

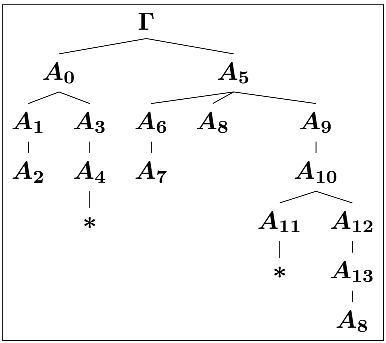
- A set of **set** $\times$ **set**-rules R is a basis for $\rhd_R$, the smallest **set** $\times$ **set**-cr containing R.
- A set of **set** $\times$ **fmla**-rules R is a basis for $\vdash_R$, the smallest **set** $\times$ **fmla**-cr containing R.

Multiple-conclusion calculi and tree-proofs

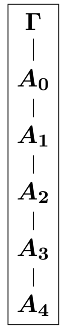
A *calculus* is a set of rules (schema) $R \subseteq \wp(L) \times \wp(L)$.

Proofs can be arboreal as rules with a conclusion set with more than a formula impose branching (case split).

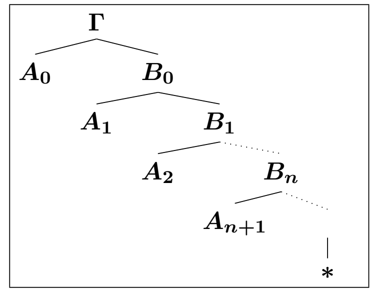
$\Gamma \rhd_R A_2, A_7, A_8$



$\Gamma \rhd_R A_4$



$\Gamma \rhd_R \{A_i : i \in \mathbb{N}\}$



Axiomatization as basis for the logic

$\rhd_R$ is the smallest **set** $\times$ **set**-cr containing R,

R is a proper basis for $\rhd_R$

Also here, **set** $\times$ **fmla**-axiomatizations are particular cases of **set** $\times$ **set**-axiomatizations.

If R are all **set** $\times$ **fmla** then $\rhd_R = \rhd_{\vdash_R}$.

Posetal categories $\mathbf{Sing}$ and $\mathbf{Mult}$

$\mathbf{Mult}$ Objects: $\langle \Sigma, \rhd \rangle$ where $\rhd$ is a **set** $\times$ **set**-cr

Morphisms: $\langle \Sigma_1, \rhd_1 \rangle \sqsubseteq \langle \Sigma_2, \rhd_2 \rangle$ if $\Sigma_1 \subseteq \Sigma_2$ and $\rhd_1 \subseteq \rhd_2$

$\mathbf{Sing}$ Objects: $\langle \Sigma, \vdash \rangle$ where $\vdash$ is a **set** $\times$ **fmla**-cr

Morphisms: $\langle \Sigma_1, \vdash_1 \rangle \sqsubseteq \langle \Sigma_2, \vdash_2 \rangle$ if $\Sigma_1 \subseteq \Sigma_2$ and $\vdash_1 \subseteq \vdash_2$

Facts:

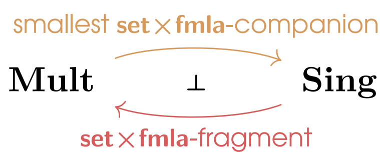
- Both are complete lattices.
- $\mathbf{Sing}$ is embeddable in $\mathbf{Mult}$ by sending $\langle \Sigma, \vdash \rangle$ to $\langle \Sigma, \rhd_\vdash \rangle$

  where $\rhd_\vdash$ is the smallest **set** $\times$ **set**-cr such that $\vdash \subseteq \rhd$.

  That is,

  $$\Gamma \rhd_\vdash \Delta \text{ iff there is } \delta \in \Delta \text{ such that } \Gamma \vdash \delta$$

- $\mathbf{Sing}$ is a full reflective subcategory of $\mathbf{Mult}$



Joins in $\mathbf{Mult}$ and $\mathbf{Sing}$

Being complete lattices both $\mathbf{Mult}$ and $\mathbf{Sing}$ bot have joins.
Given two logics $\langle \Sigma_1, \propto_1 \rangle$ and $\langle \Sigma_2, \propto_2 \rangle$ of the same type,
their join is

$$\langle \Sigma_1, \propto_1 \rangle \sqcup \langle \Sigma_2, \propto_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, \propto_1 \bullet \propto_2 \rangle$$

where $\propto_1 \bullet \propto_2$ is the smallest cr of the same type over $L_{\Sigma_1 \cup \Sigma_2}(P)$ containing $\propto_1$ and $\propto_2$.

Fact:

For sets of **set** $\times$ **set**-rules R_1 and R_2

$$\rhd_{R_1} \bullet \rhd_{R_2} = \rhd_{R_1 \cup R_2}$$

For sets of **set** $\times$ **fmla**-rules R_1 and R_2

$$\vdash_{R_1} \bullet \vdash_{R_2} = \vdash_{R_1 \cup R_2}$$

That is,
the join of two logics is axiomatized by joining axiomatizations for each

Examples of combining logics by joining their calculi

- Language extensions

Adding new connectives to a logic without imposing anything about them

Given $\rhd$ and $\vdash$ over $\mathbf{\Sigma_0} \subseteq \mathbf{\Sigma}$ let $\rhd^{\mathbf{\Sigma}}$

$\mathbf{\Gamma} \rhd^{\mathbf{\Sigma}} \mathbf{\Delta}$ iff

$\mathbf{\Gamma_0} \rhd \mathbf{\Delta_0}$ for some $\mathbf{\Gamma_0} \subseteq L_{\mathbf{\Sigma_0}}(P)$, $\mathbf{\Delta_0} \subseteq L_{\mathbf{\Sigma_0}}(P)$, $\sigma : P \to L_{\mathbf{\Sigma}}(P)$ with $\mathbf{\Gamma_0^\sigma} \subseteq \mathbf{\Gamma}$, $\mathbf{\Delta_0^\sigma} \subseteq \mathbf{\Delta}$

$$\langle \mathbf{\Sigma_0}, \rhd^{\mathbf{\Sigma}} \rangle = \langle \mathbf{\Sigma_0}, \rhd \rangle \bullet \langle \mathbf{\Sigma}, \rhd_{\textbf{norules}} \rangle$$

Examples of combining logics by joining their calculi

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Given $\rhd$ and $\vdash$ over $\Sigma_0 \subseteq \Sigma$ let $\rhd^{\Sigma}$

$\Gamma \rhd^{\Sigma} \Delta$ iff

$\Gamma_0 \rhd \Delta_0$ for some $\Gamma_0 \subseteq L_{\Sigma_0}(P)$, $\Delta_0 \subseteq L_{\Sigma_0}(P)$, $\sigma : P \to L_{\Sigma}(P)$ with $\Gamma_0^{\sigma} \subseteq \Gamma$, $\Delta_0^{\sigma} \subseteq \Delta$

$$\langle \Sigma_0, \rhd^{\Sigma} \rangle = \langle \Sigma_0, \rhd \rangle \bullet \langle \Sigma, \rhd_{\textsf{norules}} \rangle$$

- Combining classical AND and OR

Let $R_{\wedge\vee}$ be formed by the **set** $\times$ **set**-rules

$$\frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q} \qquad \frac{p}{p \vee q} \qquad \frac{p \vee p}{p}$$

$$\frac{p \wedge q}{q} \qquad \frac{p \quad q}{p \wedge q} \qquad \frac{p \vee q}{q \vee p} \qquad \frac{p \vee (q \vee r)}{(p \vee r) \vee q}$$

Examples of combining logics by joining their calculi

- Language extensions

Adding new connectives to a logic without imposing anything about them

Given $\rhd$ and $\vdash$ over $\Sigma_0 \subseteq \Sigma$ let $\rhd^\Sigma$

$\Gamma \rhd^\Sigma \Delta$ iff

$\Gamma_0 \rhd \Delta_0$ for some $\Gamma_0 \subseteq L_{\Sigma_0}(P)$, $\Delta_0 \subseteq L_{\Sigma_0}(P)$, $\sigma : P \to L_\Sigma(P)$ with $\Gamma_0^\sigma \subseteq \Gamma$, $\Delta_0^\sigma \subseteq \Delta$

$$\langle \Sigma_0, \rhd^\Sigma \rangle = \langle \Sigma_0, \rhd \rangle \bullet \langle \Sigma, \rhd_{\textsf{norules}} \rangle$$

- Combining classical AND and OR

Let $R_{\wedge\vee}$ be formed by the **set** $\times$ **set**-rules

$$\frac{p \wedge q}{p} \qquad \frac{p \wedge q}{q} \qquad \frac{p}{p \vee q} \qquad \frac{p \vee p}{p}$$

$$\frac{p \wedge q}{q} \qquad \frac{p \quad q}{p \wedge q} \qquad \frac{p \vee q}{q \vee p} \qquad \frac{p \vee (q \vee r)}{(p \vee r) \vee q}$$

- Fusion of modal logics

Seminal example and well understood via gluing Kripke frames for each of the combined logic.

Our initial <u>motivation</u> for considering PNmatrices was the difficulty in combining two given semantics to capture the effect of joining axiomatizations

Starting point: Logical matrices

Given signature $\Sigma = \{\Sigma\}_{n\in\mathbb{N}}$ and fixed $L = L_\Sigma(P)$

Logical matrix $\mathbb{M} = \langle V, \cdot_{\mathbb{M}}, \mathbf{D}\rangle$

where $\langle V, \cdot_{\mathbb{M}}\rangle$ is an algebra of truth-values

set endowed with operations $©_{\mathbb{M}} : V^n \to V$ for $© \in \Sigma^{(n)}$

$\mathbf{D} \subseteq V$ is the set of designated elements corresponding to 1

$\mathrm{Val}(\mathbb{M})$ Valuations over $\mathbb{M}$ are $v : L_\Sigma(P) \to V$ satisfying

$$v(©(\varphi_1, \ldots, \varphi_k)) = ©_{\mathbb{M}}(v(\varphi_1), \ldots, v(\varphi_k))$$

$$\Gamma \rhd_{\mathbb{M}} \Delta$$

iff

for every v over $\mathbb{M}$, $v(\Gamma) \subseteq D$ implies $v(\Delta) \cap D \neq \emptyset$.

Let $\vdash_{\mathbb{M}} = \vdash_{\rhd_{\mathbb{M}}}$.

Finite matrices $\mathbb{M}$ induce locally tabular logics, that is, $L_\Sigma(p_1, \ldots, p_k)/\dashv\vdash$ is finite.

Note that there is no finite matrix $\mathbb{M}$ such that $\rhd_{\mathbb{M}} = \rhd_{R_{\text{norules}}}$ nor $\vdash_{\mathbb{M}} = \vdash_{R_{\text{norules}}}$!

As $L_\Sigma(p_1, \ldots, p_k)/\dashv\vdash_{R_{\text{norules}}} = L_\Sigma(p_1, \ldots, p_k)$ is infinite

Extending truth-functionality: non-determinism and partiality

A Σ-PNmatrix is a tuple $\mathbb{M} = \langle V, \cdot_{\mathbb{M}}, D \rangle$

- V is a non-empty set (of *truth-values*)
- $D \subseteq V$ (the set of *designated* truth-vales)
- $©_{\mathbb{M}} : V^n \to \wp(V)$ for each $c \in \Sigma^{(n)}$
  from Baaz, Lahav & Zamansky's
  'Finite-valued semantics for canonical labelled calculi', JAR 2013

Particular cases:

Total and deterministic: Matrix If $©_{\mathbb{M}} : V^n \to \{\{a\} : a \in V\}$

Total: Nmatrix If $©_{\mathbb{M}} : V^n \to \wp(V) \setminus \{\emptyset\}$
from Avron & Lev 2005
'Non-deterministic multiple-valued structures', JAR 2013

Deterministic: Pmatrix If $©_{\mathbb{M}} : V^n \to \{\{a\} : a \in V\} \cup \{\emptyset\}$

Logics of PNmatrices

A Σ-PNmatrix is a tuple $\mathbb{M} = \langle V, \cdot_{\mathbb{M}}, D \rangle$

- V is a non-empty set (of *truth-values*)
- $D \subseteq V$ (the set of *designated* truth-vales)
- $©_{\mathbb{M}} : V^n \to \wp(V)$ for each $c \in \Sigma^{(n)}$

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$\mathrm{Val}(\mathbb{M})$ Valuations over $\mathbb{M}$ are $v : L_\Sigma(P) \to V$ satisfying

$$v(©(\varphi_1, \ldots, \varphi_k)) \in ©_{\mathbb{M}}(v(\varphi_1), \ldots, v(\varphi_k))$$

$$\Gamma \rhd_{\mathbb{M}} \Delta$$

iff

for every v over $\mathbb{M}$, $v(\Gamma) \subseteq D$ implies $v(\Delta) \cap D \neq \emptyset$.

- non-determinism gives a menu of possibilities for extending the formulas, valuations are not determined by the values over the variables!
- valuations live inside (total) subNmatrices, partiality forbids valuations combining incompatible elements
- logics of finite PNmatrices are not necessarily locally tabular

PNmatrices are nice!

- Almost(!) every logic can be characterized by a single PNmatrix
  enough for signature to contain a connective of arity > 1
- Natural semantics for logical strengthenings and combined logics
- Many non-finitely valued logics have finite PNsemantics
- Logics of finite PNmatrices are still finitary, **SAT** in NP, decision in coNP
- Effective bridge with well behaved proof-theory: logics of finite PNmatrices still can be axiomatized by finite analytical $\textbf{set} \times \textbf{set}$-calculi.

Some 2-valued Nmatrices you should know

None of the logics induced by the following Nmatrices is induced by a finite matrix (or even by a finite set of finite matrices.

$\mathbb{M}_{\text{free}}$

$©_{\text{free}}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0,1}$	$\mathbf{0,1}$
$\mathbf{1}$	$\mathbf{0,1}$	$\mathbf{0,1}$

$\rhd_{\mathbb{M}_{\text{free}}}$ is axiomatized by the emptyset of rules

$\mathbb{M}_{\text{mp}}$

$\to_{\text{mp}}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0,1}$	$\mathbf{0,1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0,1}$

$\rhd_{\mathbb{M}_{\text{mp}}}$ is axiomatized by modus ponens $\frac{p\,,\,p\to q}{q}$

$\mathbb{M}_{\text{sq}}$

	$\mathbf{0}$	$\mathbf{1}$
$\mathbb{M}_{\text{sq}}$	$\mathbf{0,1}$	$\mathbf{1}$

$\rhd_{\mathbb{M}_{\text{sq}}}$ is axiomatized by $\Box$-generalization $\frac{p}{\Box p}$

Non-determinism easily captures language extensions

Adding new connectives to a logic without imposing anything on them

Given Σ_0-PNmatrix $\mathbb{M} = \langle V, \cdot_{\mathbb{M}}, D\rangle$ let $\mathbb{M}^{\Sigma} = \langle V, \cdot_{\mathbb{M}^{\Sigma}}, D\rangle$ with

$$©(a_1, \ldots, a_k) = \begin{cases} ©_{\mathbb{M}}(a_1, \ldots, a_k) & \text{if } © \in \Sigma_0 \\ V & \text{otherwise} \end{cases}$$

Facts:

- $\rhd_{\mathbb{M}^{\Sigma}} = \rhd_{\mathbb{M}}^{\Sigma}$ and $\vdash_{\mathbb{M}^{\Sigma}} = \vdash_{\mathbb{M}}^{\Sigma}$
- If general, if $\Sigma \setminus \Sigma_0$ contains a 0-ary connective then there is no single matrix characterizing $\rhd^{\Sigma}$ or $\vdash^{\Sigma}$
- If general, if $\Sigma \setminus \Sigma_0$ contains a n-ary connective with $n > 0$ then there is no finite set of finite matrices characterizing $\rhd^{\Sigma}$ or $\vdash^{\Sigma}$

Adding axioms

There is a general recipe that generates semantics for axiomatic extensions by pre-images by strict morphisms of the original semantics (or rexpansions), yielding

- a denumerable semantics (but quite syntactic) for axiomatic extensions of logics with denumerable PNmatrix semantics, including intuitionistic and every modal logics (remember that modus ponnens and generalization can be captured by a 2-valued Nmatrix)
- a finiteness preserving semantics for a wide range of base logics and axioms satisfying certain shapes

Like the example in the first slide:

$\mathbb{B} = \langle\{0,1\},\{1\},\cdot_{\mathbb{B}}\rangle$

$\to_{\mathbb{B}}$	0	1
0	1	1
1	0	1

impose $p\to(\neg p\to\neg q)$ ⟶

$\mathbb{B}_{\mathsf{Ax}} = \langle\{00,01,10,11\},\{10,11\},\cdot_{\mathbb{B}_{\mathsf{Ax}}}\rangle$

$\to_{\mathbb{B}_{\mathsf{Ax}}}$	00	01	10	11
00	10	10	10	$\emptyset$
01	10	10, 11	10	11
10	00, 01	00, 01	10	$\emptyset$
11	$\emptyset$	01	$\emptyset$	11

	$\neg_{\mathbb{B}_{\mathsf{Ax}}}$
00	00, 01
01	10, 11
10	00, 01
11	11

Finite PNmatrices help in detecting low complexity logics

A logic decidable in **PTIME**

When apply to the following Nmatrix the algorithm generating analytical **set** $\times$ **set-** axiomatization we can observe that that this logic is decidable in **PTIME** since the generated rules are all of type **set** $\times$ **fmla** (no branching needed)

$\wedge_{\mathbb{S}}$	f	$\bot$	$\top$	t
f	f	f	f	f
$\bot$	f	$f, \bot$	f	$f, \bot$
$\top$	f	f	$\top$	$\top$
t	f	$f, \bot$	$\top$	$t, \top$

$\vee_{\mathbb{S}}$	f	$\bot$	$\top$	t
f	$f, \top$	$t, \bot$	$\top$	t
$\bot$	$t, \bot$	$t, \bot$	t	t
$\top$	$\top$	t	$\top$	t
t	t	t	t	t

	$\neg_{\mathbb{S}}$
f	t
$\bot$	$\bot$
$\top$	$\top$
t	f

$$\frac{p\,,\,q}{p \wedge q}\,r_1 \qquad \frac{p \wedge q}{p}\,r_2 \qquad \frac{p \wedge q}{q}\,r_3 \qquad \frac{\neg p}{\neg(p \wedge q)}\,r_4 \qquad \frac{\neg q}{\neg(p \wedge q)}\,r_5$$

$$\frac{p}{p \vee q}\,r_6 \qquad \frac{q}{p \vee q}\,r_7 \qquad \frac{\neg(p \vee q)}{\neg p}\,r_8 \qquad \frac{\neg(p \vee q)}{\neg q}\,r_9 \qquad \frac{\neg p\,,\,\neg q}{\neg(p \vee q)}\,r_{10}$$

$$\frac{p}{\neg\neg p}\,r_{11} \qquad \frac{\neg\neg p}{p}\,r_{12}$$

This Nmatrix was introduced in Avron&Ben-Naim&Konikowska (2007) modelling the reasoning of a processor which collects partial information from different classical sources and it was previously unknown to be of low complexity.

Categories of PNmatrices $\mathbf{PNmatr}$ and $\mathbf{PNmatr}^\flat$

A function $f : V_1 \to V_2$ is a strict morphism between $\mathbb{M}_1 = \langle \Sigma_1, \cdot_{\mathbb{M}}, D_1 \rangle$ and $\mathbb{M}_2 = \langle \Sigma_2, \cdot_{\mathbb{M}}, D_2 \rangle$ if $\Sigma_2 \subseteq \Sigma_1$ and satisfies $f^{-1}(D_2) = D_1$ and for $© \in \Sigma_2^n$,

$$f(©_{\mathbb{M}_1}(x_1, \ldots, x_n)) \subseteq ©_{\mathbb{M}_2}(f(x_1), \ldots, f(x_n))$$

This extends the notion of strict morphisms for matrices where one demands $f(©(x_1, \ldots, x_n)) = ©_{\mathbb{M}}(f(x_1), \ldots, f(x_n))$

$\mathbf{PNmatr}$:

Objects: $\langle \Sigma, \mathbb{M} \rangle$ with $\mathbb{M}$ a PNmatrix over Σ

Morphisms: strict morphisms between PNmatrices

$\mathbf{PNmatr}^\flat$:

Objects: $\langle \Sigma, \mathbb{M} \rangle$ with $\mathbb{M}$ a PNmatrix over Σ

Morphisms: $\langle \Sigma_1, \mathbb{M}_1 \rangle \sqsubseteq \langle \Sigma_2, \mathbb{M}_2 \rangle$ iff $\Sigma_2 \subseteq \Sigma_1$ and there is some strict morphism between $\mathbb{M}_1$ and $\mathbb{M}_2$. Equivalently, if $\mathbb{M}_1$ is a rexpansion of $\mathbb{M}_2$ (Avron 2020)

Facts:

- $\mathbf{PNmatr}^\flat$ is a posetal category
- Q transforms products in meets and coproducts in joins

Saturation and the ω-power

We say a PNmatrix $\mathbb{M}$ is saturated whenever $\rhd_{\mathbb{M}} = \rhd_{\vdash_{\mathbb{M}}}$, that is, whenever $\mathbf{\Gamma} \rhd_{\mathbb{M}} \mathbf{\Delta}$ iff there is $\boldsymbol{\delta} \in \mathbf{\Delta}$ such that $\mathbf{\Gamma} \vdash_{\mathbb{M}} \boldsymbol{\delta}$.

Every sound **set** $\times$ **set**-rule can be refined to a sound **set** $\times$ **fmla**-rule.

Example: The 2-valued Nmatrices $\mathbb{M}_{\mathsf{free}}$, $\mathbb{M}_{\mathbf{mp}}$, $\mathbb{M}_{\mathbf{sq}}$ and the 4-valued are all saturated

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Every sound $\mathbf{set} \times \mathbf{set}$-rule can be refined to a sound $\mathbf{set} \times \mathbf{fmla}$-rule.

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Let $\mathbf{SPNmatr}$ and $\mathbf{SPNmatr}^{\flat}$ the full subcategories of $\mathbf{PNmatr}$ and $\mathbf{PNmatr}^{\flat}$ where the objects are restricted to saturated PNmatrices.

Saturation and the ω-power

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Every sound **set** $\times$ **set**-rule can be refined to a sound **set** $\times$ **fmla**-rule.

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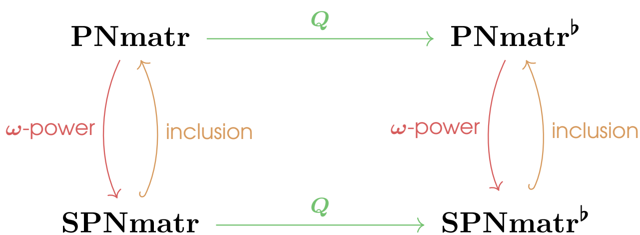


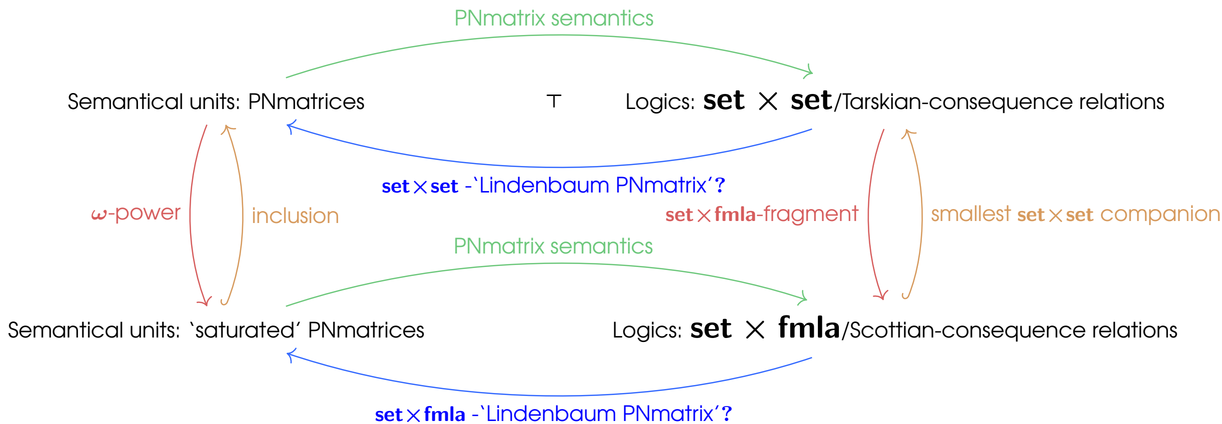
ω-power of $\mathbb{M}$ be $\mathbb{M}^\omega = \langle V^\omega, \cdot_\omega, D^\omega \rangle$ with

$$©_\omega(s_1, \dots, s_k) = \{s \in V^\omega : s(i) \in ©_{\mathbb{M}}(s_1(i), \dots, s_k(i))\}$$

Facts:

- We can always saturate a given PNmatrix: $\vdash_{\mathbb{M}} = \vdash_{\mathbb{M}^\omega}$ (not unique, sometimes enough finite power)
- By doing so we characterize the smallest **set** $\times$ **set**-companion: $\rhd_{\mathbb{M}^\omega} = \rhd_{\vdash_{\mathbb{M}}}$

From logics to PNmatrices: Lindenbaum PNmatrix?



Well...

Partiality allows for a badly behaved sum

Let $\mathcal{M} = \{\langle \Sigma, \mathbb{M}_i \rangle : i \in I\}$ be a set of PNmatrices, each $\mathbb{M}_i = \langle V_i, D_i, \cdot_{\mathbb{M}_i} \rangle$.

The sum of $\mathcal{M}$ is the PNmatrix $(\Sigma, \oplus\mathcal{M})$ where $\oplus\mathcal{M} = \langle V, D, \cdot_{\oplus} \rangle$ and

$$V = \bigcup_{i \in I} (\{i\} \times V_i)$$

$$D = \bigcup_{i \in I} (\{i\} \times D_i)$$

$$©_{\oplus}((i_1, x_1), \ldots, (i_n, x_n)) = \begin{cases} \{i\} \times ©_{\mathbb{M}_i}(x_1, \ldots, x_n)) & \text{if } i = i_1 = \cdots = i_n \\ \emptyset & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}_0$ and $c \in \Sigma^{(n)}$.

$(\Sigma, \oplus\mathcal{M})$ is a coproduct of $\mathcal{M}$ in in all the introduced PNmatrix categories $\mathbf{PNmatr}$, $\mathbf{PNmatr}^{\flat}$, $\mathbf{SPNmatr}$ and $\mathbf{SPNmatr}^{\flat}$.

Hence,

$$\bigcap_{i \in I} \mathrm{Mult}(\mathbb{M}_i) \subseteq \mathrm{Mult}(\oplus\mathcal{M})$$

Perhaps surprisingly, however, it may happen that $\mathrm{Mult}(\oplus\mathcal{M}) \neq \bigcap_{i \in I} \mathrm{Mult}(\mathbb{M}_i)$.

A sufficient condition for the equality to hold is that the Σ contains at least a connective with arity > 1.

In general we only have that $\mathrm{Mult}(\oplus\mathcal{M})$ is the smallest logic given by a single PNmatrix that contains all the logics $\rhd \mathbb{M}_i$.

Partiality allows for gathering the Lindenbaum bundle into a Pmatrix

For $\mathbf{\Gamma} \subseteq \boldsymbol{L}_{\mathbf{\Sigma}}(\boldsymbol{P})$, let $\mathbb{M}_{\mathbf{\Gamma}} = \langle \boldsymbol{L}_{\mathbf{\Sigma}}(\boldsymbol{P}), \cdot, \mathbf{\Gamma} \rangle$.

Lindenbaum bundle

$\mathrm{Lind}^{\textbf{mult}}(\langle \mathbf{\Sigma}, \rhd \rangle) = \{\mathbb{M}_{\mathbf{\Gamma}} : \mathbf{\Gamma} \not\rhd (\boldsymbol{L}_{\mathbf{\Sigma}}(\boldsymbol{P}) \setminus \mathbf{\Gamma})\}$ Maximal **set** $\times$ **set**-theories

$\mathrm{Lind}^{\textbf{sing}}(\langle \mathbf{\Sigma}, \vdash \rangle) = \{\mathbb{M}_{\mathbf{\Gamma}} : \mathbf{\Gamma} = \mathbf{\Gamma}^{\vdash} \neq \boldsymbol{L}_{\mathbf{\Sigma}}(\boldsymbol{P})\}$ All **set** $\times$ **fmla**-theories

Lindenbaum Pmatrix

Let $\mathrm{Lind}^{m}_{\oplus} : \mathbf{Mult} \to \mathbf{PNmatr}^{\flat}$

$$\mathrm{Lind}^{\textbf{mult}}_{\oplus}(\langle \mathbf{\Sigma}, \rhd \rangle) := \bigoplus \mathrm{Lind}^{\textbf{mult}}(\langle \mathbf{\Sigma}, \rhd \rangle)$$

and $\mathrm{Lind}^{s}_{\oplus} : \mathbf{Sing} \to \mathbf{SPNmatr}^{\flat}$

$$\mathrm{Lind}^{\textbf{sing}}_{\oplus}(\langle \mathbf{\Sigma}, \vdash \rangle) := \bigoplus \mathrm{Lind}^{\textbf{sing}}(\langle \mathbf{\Sigma}, \vdash \rangle)$$

Galois connection between $\mathrm{PNmatr}^\flat$ and $\mathrm{Mult}^{\mathrm{op}}$

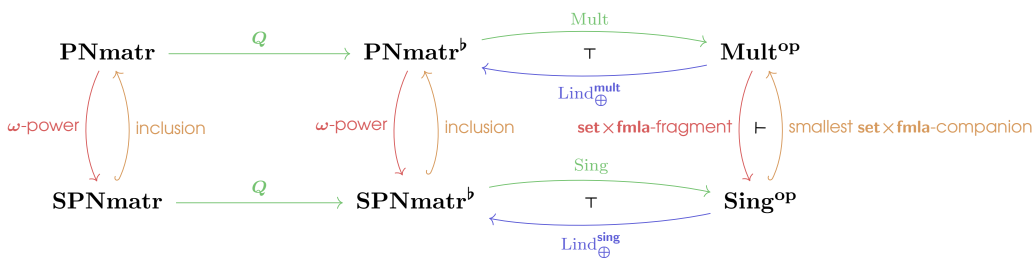
Consider the functors, in this case, also lattice morphisms

$\mathrm{Mult} : \mathbf{PNmatr}^\flat \to \mathbf{Mult}$ such that $\mathrm{Mult}(\langle \boldsymbol{\Sigma}, \mathbb{M}\rangle) = \langle \boldsymbol{\Sigma}, \rhd_{\mathbb{M}}\rangle$

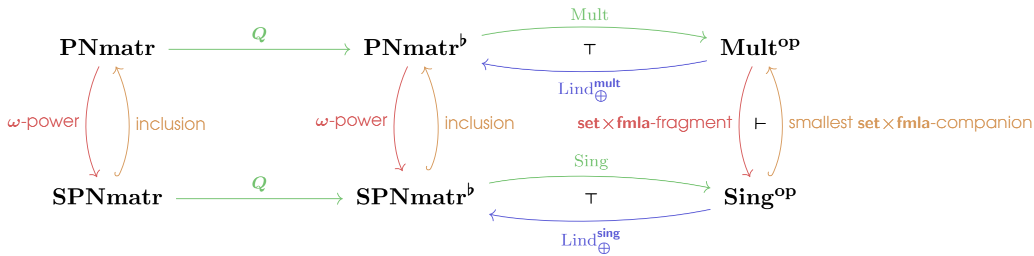
$\mathrm{Sing} : \mathbf{SPNmatr}^\flat \to \mathbf{Sing}$ such that $\mathrm{Sing}(\langle \boldsymbol{\Sigma}, \mathbb{M}\rangle) = \langle \boldsymbol{\Sigma}, \vdash_{\mathbb{M}}\rangle$

Facts:

- $\mathrm{Lind}^{\mathsf{mult}}_{\oplus}(\langle \boldsymbol{\Sigma}, \rhd\rangle) \sqsubseteq \langle \boldsymbol{\Sigma}_0, \mathbb{M}_0\rangle$ iff $\mathrm{Mult}(\langle \boldsymbol{\Sigma}_0, \mathbb{M}_0\rangle) \sqsubseteq \langle \boldsymbol{\Sigma}, \rhd\rangle$
- $\mathrm{Lind}^{\mathsf{sing}}_{\oplus}(\langle \boldsymbol{\Sigma}, \rhd\rangle) \sqsubseteq \langle \boldsymbol{\Sigma}_0, \mathbb{M}_0\rangle$ iff $\mathrm{Sing}(\langle \boldsymbol{\Sigma}_0, \mathbb{M}_0\rangle) \sqsubseteq \langle \boldsymbol{\Sigma}, \vdash\rangle$



Can we do better?



- Is there Adjunction between **PNmatr** and **Mult**? How to associate a logic with a PNmatrix such that there is a unique morphism to every PNmatrix characterizing a weaker logic? ($\mathbf{PNmatr}^\flat$ dealt with unicity)

- Is the existency of strict morphisms is sufficient to detect if PNmatrices define the same logic? No! This is big change from logical matrices... can we improve on that?

Problems $\rhd_{\mathbb{M}_1} \stackrel{?}{=} \rhd_{\mathbb{M}_2}$ and $\vdash_{\mathbb{M}_1} \stackrel{?}{=} \vdash_{\mathbb{M}_2}$

Example

	$\neg_{\mathbb{M}_1}(x)$
0	1
1	0
T	$0, T$

	$\neg_{\mathbb{M}_2}(x)$
0	1
1	0
T	$1, T$

	$\neg_{\mathbb{M}_3}(x)$
0	1
1	0
T	$0, 1, T$

	$\neg_{\mathbb{M}_4}(x)$
0	1
1	0
T	$0, T$
T'	$1, T$

Facts:

- $\mathrm{BVal}(\mathbb{M}_1) = \mathrm{BVal}(\mathbb{M}_2) = \mathrm{BVal}(\mathbb{M}_3) = \mathrm{BVal}(\mathbb{M}_4)$
- $\rhd_{\mathbb{M}_1} = \rhd_{\mathbb{M}_2} = \rhd_{\mathbb{M}_3} = \rhd_{\mathbb{M}_4}$ and $\vdash_{\mathbb{M}_1} = \vdash_{\mathbb{M}_2} = \vdash_{\mathbb{M}_3} = \vdash_{\mathbb{M}_4}$
- $\mathbb{M}_1 \sqsubseteq \mathbb{M}_3$, $\mathbb{M}_2 \sqsubseteq \mathbb{M}_3$
- $\mathbb{M}_1 \not\sqsubseteq \mathbb{M}_2$, $\mathbb{M}_2 \not\sqsubseteq \mathbb{M}_1$ and $\mathbb{M}_3 \not\sqsubseteq \mathbb{M}_4$
- $\mathbb{M}_4 \sqsubseteq \mathbb{M}_3$ and $\mathbb{M}_3$ is a quotient of $\mathbb{M}_4$.

Given arbitrary finite (P)Nmatrices the problem $\vdash_{\mathbb{M}_1} \stackrel{?}{=} \vdash_{\mathbb{M}_2}$ is undecidable.
In the multiple-conclusion setting it is still open but we suspect that the same holds for deciding $\rhd_{\mathbb{M}_1} = \rhd_{\mathbb{M}_2}$.

What changes regarding strict morphisms and quotients

Over matrices

- Kernels of strict morphisms between matrices are congruences compatible with the set of designated elements and surjective strict morphisms (and quotients) preserve the logic (both single and multiple)

- For finite reduced Σ-matrices $\mathbb{M}_1$ and $\mathbb{M}_2$,

  $\rhd_{\mathbb{M}_1} = \rhd_{\mathbb{M}_2}$ IFF there are strict morphisms $f_{12} : \mathbb{M}_1 \to \mathbb{M}_2$ and $f_{21} : \mathbb{M}_2 \to \mathbb{M}_1$ (Shoesmith and Smiley 1978)

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Over PNmatrices

- Any quotient of a PNmatrix by an equivalence relation compatible with the set of designated values is still a PNmatrix and induces a strict morphism (and vice-versa)
- A strict (surjective or not) morphism $f : \mathbb{M}_1 \to \mathbb{M}_2$ only implies that $\rhd_{\mathbb{M}_2} \subseteq \rhd_{\mathbb{M}_1}$
- Strict morphisms (and quotients) of PNmatrices may generate stronger logics
- Of course that if there are strict morphisms $f_{12} : \mathbb{M}_1 \to \mathbb{M}_2$ and $f_{21} : \mathbb{M}_2 \to \mathbb{M}_1$ then $\rhd_{\mathbb{M}_1} = \rhd_{\mathbb{M}_2}$ but the other direction fails
- Perhaps a local explanation for $\rhd_{\mathbb{M}_1} = \rhd_{\mathbb{M}_2}$ soundness is not possible

Full circle: general semantics for combined logics

Strict product of PNmatrices

Given Σ_1- and Σ_2-PNmatrices $\mathbb{M}_1 = \langle A_1, \cdot_1, D_1 \rangle$ and $\mathbb{M}_2 = \langle A_2, \cdot_2, D_2 \rangle$,
let $U_1 = A_1 \setminus D_1$ and $U_2 = A_2 \setminus D_2$.

Their strict product is the $\Sigma_1 \cup \Sigma_2$-PNmatrix

$$\mathbb{M}_1 \star \mathbb{M}_2 = \langle A_{12}, \cdot_{12}, D_{12} \rangle$$

where

$$A_{12} = (D_1 \times D_2) \cup (U_1 \times U_2) \qquad D_{12} = D_1 \times D_2$$

$$©_{12}((a_1, b_1), \dots, (a_k, b_k)) = \begin{cases} \{(a,b) \in A_{12} : a \in ©_1(a_1, \dots, a_k)\} & \text{if } c \in \Sigma_1 \setminus \Sigma_2 \\ \{(a,b) \in A_{12} : b \in ©_2(b_1, \dots, b_k)\} & \text{if } c \in \Sigma_2 \setminus \Sigma_1 \\ \{(a,b) \in A_{12} : a \in ©_1(a_1, \dots, a_k) \\ \qquad \text{and } b \in ©_2(b_1, \dots, b_k)\} & \text{if } c \in \Sigma_1 \cap \Sigma_2 \end{cases}$$

Note that $©_{12}((a_1, b_1), \dots, (a_k, b_k)) = \emptyset$
if $©_1(a_1, \dots, a_k) \subseteq D_1$ and $©_2(a_1, \dots, a_k) \subseteq U_2$ or vice versa.

Facts about strict-product

- $\mathbb{M}_1 \star \mathbb{M}_2$ is saturated whenever $\mathbb{M}_1$ and $\mathbb{M}_2$ are
- $\langle \Sigma_1, \mathbb{M}_1 \rangle \otimes \langle \Sigma_2, \mathbb{M}_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, \mathbb{M}_1 \star \mathbb{M}_2 \rangle$ is the product in all the introduced PNmatrix categories $\mathbf{PNmatr}$, $\mathbf{PNmatr}^\flat$, $\mathbf{SPNmatr}$ and $\mathbf{SPNmatr}^\flat$.
  - $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are strict-morphisms
  - $\mathrm{Val}(\mathbb{M}_1 * \mathbb{M}_2) = \mathrm{Val}(\mathbb{M}_1^{\Sigma_1 \cup \Sigma_2}) \cap \mathrm{Val}(\mathbb{M}_2^{\Sigma_1 \cup \Sigma_2})$.
    * If $v \in \mathrm{Val}(\mathbb{M}_1 * \mathbb{M}_2)$ then $(\pi_k \circ v) \in \mathrm{Val}(\mathbb{M}_k{}^{\Sigma_1 \cup \Sigma_2})$
    * $v_1 \in \mathrm{Val}(\mathbb{M}_1^{\Sigma_1 \cup \Sigma_2})$, $v_2 \in \mathrm{Val}(\mathbb{M}_2^{\Sigma_1 \cup \Sigma_2})$, and $v_1(\varphi) \in D_1$ iff $v_2(\varphi) \in D_2$ for every $A \in L_{\Sigma_1 \cup \Sigma_2}(P)$, then $v_1 * v_2 \in \mathrm{Val}(\mathbb{M}_1 * \mathbb{M}_2)$ with $v_1 * v_2(\varphi) = (v_1(\varphi), v_2(\varphi))$

Modular semantics for combined logics by joining calculi

- Product on the semantical side, coproduct of logics
- $\rhd_{\mathbb{M}_1} \sqcup \rhd_{\mathbb{M}_2} = \rhd_{\mathbb{M}_1 \star \mathbb{M}_2}$
- If $\mathbb{M}_1$ and $\mathbb{M}_2$ saturated then $\vdash_{\mathbb{M}_1} \sqcup \vdash_{\mathbb{M}_2} = \vdash_{\mathbb{M}_1 \star \mathbb{M}_2}$
- If either $\mathbb{M}_1$ or $\mathbb{M}_2$ not saturated it may happen that $\vdash_{\mathbb{M}_1} \sqcup \vdash_{\mathbb{M}_2} \subsetneq \vdash_{\mathbb{M}_1 \star \mathbb{M}_2}$
- In any case, $\vdash_{\mathbb{M}_1} \sqcup \vdash_{\mathbb{M}_2} = \vdash_{\mathbb{M}_1^\omega \star \mathbb{M}_2^\omega}$

Back to combining AND and OR

Let $\mathbf{2}_{\wedge}$:

$\tilde{\wedge}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$

$\mathbf{2}_{\vee}$:

$\tilde{\vee}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

Back to combining AND and OR

Let $\mathbf{2}_{\wedge}$:

$\tilde{\wedge}$	0	1
0	0	0
1	0	1

$\mathbf{2}_{\vee}$:

$\tilde{\vee}$	0	1
0	0	1
1	1	1

In **set** $\times$ **set**:

$\mathbf{2}_{\wedge} \star \mathbf{2}_{\vee} = \mathbf{2}_{\wedge\vee}$ is the $\wedge\vee$-fragment of classical Boolean matrix and indeed the rules $\frac{p \wedge q}{p}$ $\frac{p \wedge q}{q}$ $\frac{p \quad q}{p \wedge q}$ $\frac{p}{p \vee q}$ $\frac{q}{p \vee q}$ $\frac{p \vee q}{p \, , \, q}$ axiomatize $\rhd_{\mathbf{2}_{\wedge\vee}}$.

Back to combining AND and OR

Let $\mathbf{2}_{\wedge}:$

$\tilde{\wedge}$	0	1
0	0	0
1	0	1

$\mathbf{2}_{\vee}:$

$\tilde{\vee}$	0	1
0	0	1
1	1	1

In **set** $\times$ **set**:

$\mathbf{2}_{\wedge} \star \mathbf{2}_{\vee} = \mathbf{2}_{\wedge\vee}$ is the $\wedge\vee$-fragment of classical Boolean matrix and indeed the rules $\frac{p \wedge q}{p}$ $\frac{p \wedge q}{q}$ $\frac{p \quad q}{p \wedge q}$ $\frac{p}{p \vee q}$ $\frac{q}{p \vee q}$ $\frac{p \vee q}{p\,,\,q}$ axiomatize $\rhd_{\mathbf{2}_{\wedge\vee}}$.

In **set** $\times$ **fmla**:

$\mathbf{2}_{\wedge}$ is saturated but $\mathbf{2}_{\vee}$ is not. $p \vee q \rhd_{\mathbf{2}_{\vee}} p, q$ but $p \vee q \not\vdash_{\mathbf{2}_{\vee}} p$ and $p \vee q \not\vdash_{\mathbf{2}_{\vee}} q$

$$\vdash_{\wedge\vee^{\omega}} = \vdash_{\mathbf{2}_{\wedge} \star \mathbf{2}_{\vee}^{\omega}} \subsetneq \vdash_{\mathbf{2}_{\wedge\vee}} \text{ and } \mathbf{2}_{\wedge\vee} \cong \mathbf{2}_{\wedge} \star \mathbf{2}_{\vee}^{\omega} \text{ where}$$

$\mathbf{2}_{\wedge\vee} = \langle \wp(\mathbb{N}), \cdot_{\#}, \{\mathbb{N}\} \rangle$ with

$$X \vee_{\#} Y = X \cup Y \text{ and } X \wedge_{\#} Y = \begin{cases} \mathbb{N} & \text{if } X = Y = \mathbb{N} \\ \wp(\mathbb{N}) & \text{otherwise} \end{cases}$$

Fact:

Classical logic can be axiomatized joining axiomatizations for each of fragments with a single connective in **set** $\times$ **set** but not in **set** $\times$ **fmla**.

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